827

PLASTIC BENDING OF CIRCULAR PLATES SYMMETRICALLY LOADED

BY

KENNETH R. MERCKX

TECHNICAL REPORT NO. 16

PREPARED UNDER CONTRACT N6ONR-251 TASK ORDER 11
(NR-064-240)
FOR
OFFICE OF NAVAL RESEARCH

DIVISION OF ENGINEERING MECHANICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

JUNE, 195?

PLASTIC BENDING

OF

CIRCULAR PLATES

SYMMETRICALLY LOADED

BY

KENNETH MERCKX

TECHNICAL REPORT NO. 16

PREPARED UNDER CONTRACT N6ONR-251 TASK ORDER 11
(NR-064-240)
UNDER THE DIRECTION OF DR. ARIS PHILLIPS

DIVISION OF ENGINEERING MECHANICS
STANFORD UNIVERSITY
STANFORD, CALIFORNIA

JUNE, 1953

ABSTRACT

Because materials are often stressed into the plastic range in light weight construction, a need exists for information concerning the plastic behavior of structural members. A plate subjected to lateral loads is such a structural member. In this report, we use several general methods of plasticity to obtain information about the plastic behavior of circular plates and we suggest how these methods can be extended to rectangular plates.

In this report, the kinematic relations connecting the displacements of the neutral surface to the strains are those of the small deflection theory of bending. The stress-strain relations are those of the theory of plastic deformations (secant modulus theory). In the last chapter, the comparison of the theory of plastic flow and the theory of plastic deformations indicates that only slight differences would be found in numerical results calculated by either of these two theories.

The four approximate methods which are applied to circular plates are

1). Sokolovsky's Method,
2). Iteration Method,
3). Potential Energy Method,
4). Complementary Potential Energy Method.

Sokolovsky reduces the equations relating the moments, curvatures, and loads of the plate to two simultaneous first order non-linear differential equations. These equations are then solved by numerical integration. The iteration method, developed by Ilyushin, is adapted for the bending of circular plates. In this method we separate a non-linear second order differential equation into a linear portion and a non-linear portion. The effect of the non-linear portion on the solution is found by the iteration procedure. The principle of minimum potential energy is used to estimate the circumferential curvature of the plate by means of functions containing arbitrary parameters. These parameters are evaluated by Galerkin's Method. The principle of minimum complementary potential energy is used to estimate the radial bending moment in the plate by a similar process.

Numerical results are obtained by all four methods for a simply supported uniformly loaded plate for a material where the second stress invariant S is related to the second strain invariant E by

$$S = 2G(1 - \lambda E^2)E.$$

A comparison of the results of the above methods is given in the last chapter of this report. Also, in this chapter, the last three of the above approximate methods are developed for rectangular plates, but no numerical calculations are made.

CONTENTS

CHAPTER I

INTRODUCTION

Section

CHAPTER II

THE METHOD OF SOKOLOVSKY

CHAPTER III

THE ITERATION METHOD

CHAPTER IV

THE APPLICATION OF POTENTIAL ENERGY

CHAPTER V

THE APPLICATION OF COMPLEMENTARY POTENTIAL ENERGY

CHAPTER VI

CONCLUSION

CHAPTER I

INTRODUCTION

1.1 The Problem Before Us.

The circular plate symmetrically loaded by lateral forces is a structural member which the practicing engineer often encounters. A schematic diagram of the problem is shown in figure 1. The circular symmetry of the loading enables the deflection and stress condition of the plate to be represented as functions dependent upon the radial distance r. Thus, the physical relationships are expressed in terms of ordinary differential equations.

This work considers the small deflection theory of plates due to lateral loads. The strains of the neutral surface and the quadratic terms in the strains due to bending are neglected. The laws of the theory of plastic deformations are used instead of elastic stress-strain relationships.

Naghdi (1) considers the large deflection theory of circular plates with stretching of the neutral surface. Gleyzal (2), Mostow (3), and Hill (4), approximate the stress condition for the large deflections of thin plates by considering only membrane stresses and strains.

1.2 Objective

For a general problem, such as the bending of a plate, several procedures are available for obtaining a solution. The preferable method, in any particular case, depends upon the accuracy and information desired. The most desirable method is one which yields the exact solution. For the plastic bending of a circular plate no such exact solution has been found. Therefore, this thesis applied several general methods of plasticity to obtain approximate solutions and a comparison is made of the accuracy and labor involved in these methods.

In the actual computation of the example problem, a dimensionless parameter appears in terms of which the range and accuracy of the various methods are indicated. The comparison of the results, obtained by the different methods of solution applied to the problem of the circular plate, should give some insight into those problems which present greater mathematical difficulties, such as the bending of rectangular plates.

1.3 The Kinematic and Equilibrium Equations for the Symmetrical Bending of Circular Plates.

This section states the kinematic and equilibrium conditions for the symmetrical bending of circular plates. A complete presentation of what is mentioned here is given by Timoshenko (5) (Chap. 3).

The following assumptions are made:

a). The defloction of the neutral surface of the plate is of such a magnitude that the quadratic terms due to bending are neglected in the strain formulae.

b). The elongation of the neutral surface and the membrane stresses are neglected.

c). Normals to the neutral surface before deflection are normal to the neutral surface in the deflected state.

d). The normal stress σ_z is assumed zero throughout the plate.

e). One additional condition is assumed due to the nature of the plastic laws. This condition is that the loading is applied proportionally and increases monotonously. Hence, only loading is considered.

In stating the kinematic relations of the plate the following notations are used.

w = deflection of the neutral surface from the unstrained to the strained condition.

$\alpha_\theta = 1/\rho_\theta$ = circumferential curvature.

$\alpha_r = 1/\rho_r$ = radial curvature

u = radial displacement

z = distance from the neutral surface.

The curvatures are expressed in terms of the deflection as

$$\alpha_\theta = -1/r \frac{dw}{dr} \quad \text{and} \quad \alpha_r = -\frac{d^2w}{dr^2}. \tag{1.1}$$

The geometry of figure 2 suggests the physical significance of α_θ and α_r. The radial radius of curvature, ρ_r, is the radius of curvature of the intersection of the neutral surface and the vertical plane containing a radius of the plate. The circumferential radius of curvature, ρ_θ, is the radius of curvature of the intersection of the neutral surface and the plane perpendicular to the tangent of the curve in the neutral surface at θ = const.

The geometry of figure 3 gives

$$u = -z \frac{dw}{dr}.$$

Due to the circular symmetry of the problem, the circumferential displacement is zero.

From known formulae for the strains in terms of displacements, the equations for the radial strain, ϵ_r, and the circumferential strain, ϵ_θ, are

$$\varepsilon_r = \frac{du}{dr} = -z\frac{d^2w}{dr^2} = z\alpha_r$$

and (1.2)

$$\varepsilon_\theta = \frac{u}{r} = -\frac{z}{r}\frac{dw}{dr} = z\alpha_\theta .$$

Another important equality derived from (1.1) is

$$\frac{d(r\alpha_\theta)}{dr} = \alpha_r . \tag{1.3}$$

The following notation is used to formulate the equilibrium equations:

$p(r)$ = lateral load per unit area.

Q_r = radial shearing force per unit length.

M_r = radial moment per unit length.

M_θ = circumferential moment per unit length.

A pictorial representation of these quantities is given in figure 4.

The definition of the moments is

$$M_r = 2\int_0^h \sigma_r\, zdz,$$

and (1.4)

$$M_\theta = 2\int_0^h \sigma_\theta\, zdz,$$

where σ_r is the radial stress and σ_θ is the circumferential stress. It has been assumed that the radial and circumferential stresses have point symmetry about $z = 0$.

Consideration of the equilibrium of the forces on the circular section of the plate shown in figure 4a yields

$$rQ_r = \int_0^r prdr. \tag{1.5}$$

The equilibrium equation is found by using the above equality and the equilibrium of the moments on the element shown in figure 4b

$$\frac{d(rM_r)}{dr} - M_\theta + \int_0^r prdr = 0. \qquad (1.6)$$

1.4 Development of the Stress-Strain Relationships.

This work is not intended to be a discussion of the relative merits of the different theories of plasticity; therefore, this section briefly presents the laws of plasticity used in this paper. A complete development of the relationships used in this section is given by Sokolovsky (6) (chap. 1). A thorough explanation of the yield condition is found in the book of Hill (7) (chap. 2). The theory of plastic deformations (secant modulus theory) is used.

The plastic stress-strain relationships are now introduced. The principal stresses are σ_r, σ_θ, and σ_z. The principal strains are ε_r, ε_θ, and ε_z.

The mean stress is

$$\sigma = 1/3\ (\sigma_r + \sigma_\theta + \sigma_z). \qquad (1.7)$$

The mean strain is

$$\varepsilon = 1/3\ (\varepsilon_r + \varepsilon_\theta + \varepsilon_z). \qquad (1.8)$$

The stress deviations are

$$s_i = \sigma_i - \sigma\ (i = r, \theta, z). \qquad (1.9)$$

The strain deviations are

$$e_i = \varepsilon_i - \varepsilon\ (i = r, \theta, z). \qquad (1.10)$$

The second stress invariant is

$$S^2 = \frac{1}{2}\ (s_r^{\,2} + s_\theta^{\,2} + s_z^{\,2}). \qquad (1.11)$$

The second strain invariant is

$$E^2 = \frac{1}{2} (e_r^2 + e_\theta^2 + e_z^2). \tag{1.12}$$

The following assumptions of the secant modulus theory are used.

a). The stress deviations are proportional to the corresponding strain deviations.

$$\frac{s_r}{e_r} = \frac{s_\theta}{e_\theta} = \frac{s_z}{e_z}$$

With the use of algebraic operations and the definitions (1.11) and (1.12), the above relation is formulated as

$$\frac{s_r}{e_r} = \frac{s_\theta}{e_\theta} = \frac{s_z}{e_z} = \frac{S}{E}. \tag{1.13}$$

b). The second stress invariant is a function of the second strain invariant. This relationship is written as

$$S = 2G \left[1 - f(E)\right] E. \tag{1.14}$$

If the relationship between S and E has point symmetry about the origin, then f(E) is an even function of E. Thus, the function, f(E), can be expressed as

$$f(E) = \sum_{n=1}^{\infty} \lambda_n E^{2n}, \tag{1.15}$$

where the λ's are constants.

c). The material is incompressible,

$$\varepsilon = \frac{1}{3} (\varepsilon_r + \varepsilon_\theta + \varepsilon_z) = 0$$

from which it follows that

$$\varepsilon_z = - (\varepsilon_\theta + \varepsilon_r). \tag{1.16}$$

This assumption greatly simplifies the mathematical formulation of the problem.

The principal stress deviations and stress invariant are obtained by using equations (1.7), (1.9), (1.11), and assumption d in section 1.3 that $\sigma_z = 0$:

$$s_r = \frac{2}{3}\left(\sigma_r - \frac{\sigma_\theta}{2}\right),\ s_\theta = \frac{2}{3}\left(\sigma_\theta - \frac{\sigma_r}{2}\right),\ s_z = -\frac{1}{3}\left(\sigma_r + \sigma_\theta\right) \tag{1.17}$$

$$S^2 = \frac{1}{3}\left(\sigma_r^{\,2} - \sigma_r\sigma_\theta + \sigma_\theta^{\,2}\right). \tag{1.18}$$

The principal strain deviations and strain invariant are obtained with the aid of equations (1.8), (1.10), (1.12), and (1.16):

$$e_r = \varepsilon_r,\ e_\theta = \varepsilon_\theta,\ e_z = \varepsilon_z = -(\varepsilon_r + \varepsilon_\theta) \tag{1.19}$$

$$E^2 = \varepsilon_r^{\,2} + \varepsilon_r\varepsilon_\theta + \varepsilon_\theta^{\,2} \tag{1.20}$$

With the expressions (1.13), (1.15), (1.17), (1.19), and (1.20), the principal stresses σ_r and σ_θ are expressed in terms of the principal strains:

$$\begin{aligned} \sigma_r &= 4G\left[1 - \sum_{n=1}^{\infty}\lambda_n\left(\varepsilon_r^{\,2} + \varepsilon_r\varepsilon_\theta + \varepsilon_\theta^{\,2}\right)^n\right]\left(\varepsilon_r + \frac{\varepsilon_\theta}{2}\right) \\ \sigma_\theta &= 4G\left[1 - \sum_{n=1}^{\infty}\lambda_n\left(\varepsilon_r^{\,2} + \varepsilon_r\varepsilon_\theta + \varepsilon_\theta^{\,2}\right)^n\right]\left(\varepsilon_\theta + \frac{\varepsilon_r}{2}\right) \end{aligned} \tag{1.21}$$

These are the stress-strain relations.

1.5 Formulation of the Basic Plate Equations.

This section presents the formulae which are used in the solution of the bending of a circular plate.

By substituting expressions (1.1) and (1.21) into equations (1.4) and integrating, we find the bending moments

$$M_r = D\,(1 - B)\,(\alpha_r + \frac{\alpha_\theta}{2})$$

$$M_\theta = D\,(1 - B)\,(\alpha_\theta + \frac{\alpha_r}{2}), \tag{1.22}$$

where

$$D = \frac{8h^3}{3} G$$

$$B = \sum_{n=1}^{\infty} \frac{3\lambda_n h^{2n}}{2n+3} (\alpha_r^2 + \alpha_r\alpha_\theta + \alpha_\theta^2)^n. \tag{1.23}$$

The integration of (1.4) is possible because the α_r and α_θ are constant for a given r by assumption c in section 1.3. The constant D is the customary expression for the flexural rigidity of a plate in elasticity. The quantity B expresses the non-linearity characteristic of the plastic stress-strain relationship.

The solution for the bending of the circular plate is found when M_r, M_θ, α_r, α_θ, and w are found as functions of r. The five relationships in (1.1), (1.3), (1.6), (1.22) are sufficient for the determination of the above functions. The combination of these five equations yields

$$\frac{d^3w}{dr^3} + \frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr} = \frac{1}{Dr}\int_0^r prdr \tag{1.24}$$

$$+ \frac{d}{dr}\left\{\left(\frac{d^2w}{dr^2} + \frac{1}{2r}\frac{dw}{dr}\right)\sum_{n=1}^{\infty}\frac{3\lambda_n h^{2n}}{2n+3}\left[\left(\frac{d^2w}{dr^2}\right)^2 + \left(\frac{d^2w}{dr^2}\right)\left(\frac{1}{r}\frac{dw}{dr}\right) + \left(\frac{1}{r}\frac{dw}{dr}\right)^2\right]^n\right\}$$

$$+ \frac{1}{2}\left[\frac{1}{r}\frac{d^2w}{dr^2} - \frac{1}{r^2}\frac{dw}{dr}\right]\sum_{n=1}^{\infty}\frac{3\lambda_n h^{2n}}{2n+3}\left[\left(\frac{d^2w}{dr^2}\right)^2 + \left(\frac{d^2w}{dr^2}\right)\left(\frac{1}{r}\frac{dw}{dr}\right) + \left(\frac{1}{r}\frac{dw}{dr}\right)^2\right]^n .$$

The values M_r, M_θ, α_r, α_θ, and w are not found by solving equation (1.24) directly. Different dependent variables are introduced in this thesis which reduce the numerical work in the various approximate methods.

Table I is compiled for future reference. The formulae in this table are (1.1), (1.3), (1.6), (1.22), and (1.23).

TABLE I

$$\alpha_\theta = -\frac{1}{r}\frac{dw}{dr} \tag{I.}$$

$$\alpha_r = \frac{d(r\alpha_\theta)}{dr} \tag{II.}$$

$$\frac{d(rM_r)}{dr} - M_\theta + \int_0^r prdr = 0 \tag{III.}$$

$$M_r = D(1 - B)\left(\alpha_r + \frac{\alpha_\theta}{2}\right),\ M_\theta = D(1 - B)\left(\alpha_\theta + \frac{\alpha_r}{2}\right) \tag{IV.}$$

$$D = \frac{8h^3}{3}G,\ B = \sum_{n=1}^{\infty}\frac{3\lambda_n h^{2n}}{2n+3}\left(\alpha_r^2 + \alpha_r\alpha_\theta + \alpha_\theta^2\right)^n \tag{V.}$$

CHAPTER II

THE METHOD OF SOKOLOVSKY

2.1 Mathematical Formulation of the Problem.

Sokolovsky (6) (chap. 14) treats the problem of the bending of a circular plate symmetrically loaded. In principle, Sokolovsky reduces formulae II, III, and IV into two simultaneous first order ordinary differential equations. By the introduction of two new variables, a magnitude factor A and an angular factor ω, which replace the curvatures α_r and α_θ, these two differential equations can be expressed in a convenient form. The variables A and ω are defined in terms of α_r and α_θ as

$$\alpha_r = -\frac{2A}{3}\sin(\omega - \pi/6), \quad \alpha_\theta = \frac{2A}{3}\sin(\omega + \pi/6). \tag{2.1}$$

With the use of (2.1), the following identity for A is formed:

$$A^2 = \alpha_r^2 + \alpha_r\alpha_\theta + \alpha_\theta^2. \tag{2.2}$$

A combination of (1.2), (1.20), and (2.2) yields

$$E^2 = z^2A^2, \tag{2.3}$$

which shows that the factor z is the proportionality constant between E and A. This same proportionality exists between the principal strains and the curvatures as seen in (1.2).

A graphical representation of ω is given in figures 5a and 5b. The normal of the plane of figure 5b makes equal angles with the three orthogonal axes α_r', α_θ', $-(\alpha_r' + \alpha_\theta')$ of figure 5a. This plane is called the octahedral plane. The axes shown in figure 5b are the projections of the axes of figure 5a. A vector with the components $\sqrt{2}/3\ \alpha_r{}'$, $\sqrt{2}/3\ \alpha_\theta{}'$, and $\sqrt{2}/3\left[-(\alpha_r{}' + \alpha_\theta{}')\right]$, has the length $2/\sqrt{3}A$ and lies in the plane of figure 5b. The vector $2/\sqrt{3}A$ lies at an angle ω, as shown in figure 5b. It has the orthogonal projections α_r and α_θ on the axes α_r and α_θ in the octahedral plane. When the proportional principal strains are substituted for the curvatures in the above graphical representation, the vector $2/\sqrt{3}A$ becomes the octahedral shearing strain.

With the substitution of (2.1) and (2.2) in IV and V of Table I, the moments are expressed as

$$M_r = D\,(1 - B)\,A\cos(\omega + \pi/6), \tag{2.4}$$

$$M_\theta = D\,(1 - B)\,A\cos(\omega - \pi/6),$$

where

$$B = \sum_{n=1}^{\infty} \frac{3\lambda_n}{2n + 1}(hA)^{2n}. \tag{2.5}$$

The substitution of (2.1) into II of Table I yields

$$r\left[\sin(\omega + \pi/6)\frac{dA}{dr} + A\cos(\omega + \pi/6)\frac{d\omega}{dr}\right] + \sqrt{3}A\sin\omega = 0. \tag{2.6}$$

Similarly, the substitution of (2.4) into II of Table I gives

$$r\left[\cos(\omega + \pi/6)\left(1 - \frac{dB}{dA}\right)\frac{dA}{dr} - (1 - B)\,A\sin(\omega + \pi/6)\frac{d\omega}{dr}\right]$$

$$- (1 - B)\,A\sin\omega + \int_0^r rp(r)\,dr/D = 0. \tag{2.7}$$

These two equations are reorganized into a more convenient form for numerical calculations when the derivatives of ω and A are separated. This separation yields

$$r\left[(1 - B) - A\frac{dB}{dA} + A\frac{dB}{dA}\sin^2(\omega + \pi/6)\right]\frac{d\omega}{dr} - \sqrt{3}A^2\frac{dB}{dA}\sin\omega\cos(\omega + \pi/6)$$

$$+ A(1 - B)\sin 2\omega - \sin(\omega + \pi/6)\int_0^r rp(r)\,dr/D = 0 \qquad (2.8)$$

$$r\left[(1 - B) - A\frac{dB}{dA} + A\frac{dB}{dA}\sin^2(\omega + \pi/6)\right]\frac{dA}{dr} + 2A(1 - B)\sin^2\omega$$

$$+ \cos(\omega + \pi/6)\int_0^r rp(r)\,dr/D = 0. \qquad (2.9)$$

Equations (2.8) and (2.9) are the desired first order ordinary differential equations. They can be solved by numerical methods when definite stress-strain laws and boundary conditions are given. When $r = 0$ and $A \neq 0$, equations (2.8) and (2.9) both yield $\omega = 0$ and A is finite. The substitution of $\omega = 0$ into (2.4) gives

$$M_r = M_\theta = \frac{\sqrt{3}}{2}D(1 - B)A.$$

Expression III Table I also yields that $M_r = M_\theta$ at the center of the plate where $r = 0$. Thus, one boundary condition is:

At the center of the plate where $r = 0$

$$M_r = M_\theta \qquad \text{or} \qquad \omega = 0. \qquad (2.10)$$

Two boundary conditions are necessary to solve equations (2.8) and (2.9). The second boundary condition is obtained by physical restrictions on the curvatures or moments at the outer edge of the plate. With the use of either equations (2.1) or (2.4) and the given boundary conditions at the outer edge of the plate, values for A or ω are determined for this outer boundary. In the next two sections, a method of solution is demonstrated by solving an example problem.

2.2 The Illustrative Example.

An illustrative example of a simply supported plate with constant pressure p is solved to indicate the method of solution for (2.8) and (2.9). The stress-strain relation (1.14) is assumed as

$$S = 2G\,(1 - \lambda E^2)\,E. \tag{2.11}$$

A graphical representation of this curve is given in figure 6 which shows a horizontal tangent at $E = E_o$. From the condition of a horizontal tangent at E_o, the constant λ is determined by

$$d\,S/dE = 2G\,(1 - 3\lambda E_o^{\,2}) = 0.$$

from which it follows

$$\lambda = 1/3\,E_o^{\,2}. \tag{2.12}$$

Two new dimensionless variables a and y are introduced to simplify the form of (2.8) and (2.9) for this example. These variables are expressed in terms of A and r as

$$a = A/A_o,\; y = (\lambda^{1/2}\,h\,p/D)^{1/2} r, \tag{2.13}$$

where the constant A_o is defined by

$$h^2 A_o^{\,2} = 1/3\,\lambda. \tag{2.14}$$

The maximum E must occur at $z = h$, and E_o is the maximum permissible strain invariant. A combination of (2.3), (2.12), and (2.14) gives $E_o^{\,2} = h^2 A_o^{\,2}$; thus, A_o can be interpreted as a measure of the maximum permissible strain condition in the plate. This interpretation limits the value of a to $0 \leq a \leq 1$. The factor B, which is given in expression (2.5), is given in terms of a as

$$B = \frac{3}{5}\lambda h^2 A^2 = 0.2a^2 \tag{2.15}$$

for the stress-strain relation (2.11). When the pressure p is constant, the load integral is

$$\int_0^r prdr = pr^2/2. \tag{2.16}$$

With the use of (2.15) and (2.16), equations (2.8) and (2.9) are expressed in terms of the variables ω, a, and y as

$$ya\left[1 - 0.6a^2 + 0.4a^2 \sin^2(\omega + \pi/6)\right]\frac{d\omega}{dy} - 0.6928\, a^3 \sin\omega \cos(\omega + \pi/6)$$

$$+ a\,(1 - 0.2a^2)\sin 2\omega - 0.866\, y^2 \sin(\omega + \pi/6) = 0 \tag{2.17}$$

$$y\left[1 - 0.6a^2 + 0.4a^2 \sin^2(\omega + \pi/6)\right]\frac{da}{dy} + 2a\,(1 - 0.2a^2)\sin^2\omega$$

$$+ 0.866\, y^2 \cos(\omega + \pi/6) = 0 \tag{2.18}$$

For a simply supported plate the radial moment is zero at the outer edge of the plate. From (2.4), this boundary condition is evaluated as:

1). At the outer edge of the plate, r = b

$$M_r = 0 \quad \text{or} \quad \omega(Y) = \pi/3 \quad \text{where} \quad Y = \left[\lambda^{1/2}\, h\, pb^2/D\right]^{1/2}, \tag{2.19}$$

2). At the center of the plate, r = 0, expression (2.10) gives

$$M_r = M_\theta \quad \text{or} \quad \omega = 0 \quad \text{at} \quad y = 0. \tag{2.20}$$

Conditions (2.19) and (2.20) determine unique solutions of (2.17) and (2.18).

A step by step numerical integration of (2.17) and (2.18) can be started when a value for "a" at y = 0 is chosen. This step by step integration is continued until $\omega = \pi/3$, and by condition (2.19) the value of Y, where $\omega = \pi/3$, determines the outer boundary of the plate. When Y is prescribed, a series of starting values for "a" are assumed. These starting values are adjusted until one gives the correct Y at $\omega_{(Y)} = \pi/3$. In the next section the numerical procedure is explained by an example where the initial conditions are $y = 0$, $\omega = 0$, and $a = 0.5$.

2.3 The Numerical Integration:

A numerical method used for computational work should have a checking procedure. When evaluation of the numerical steps becomes tedious, calculation errors arise unless an automatic check is available.

An accurate procedure satisfying the above conditions is Milne's Method. Scarborough (8) (pp 245 and 295) gives a thorough explanation of this process. In this method, the following formulae are used:

Milne's formula

$$\omega_{n+1}^{(1)} = \omega_{n-3} + \frac{4\,\Delta y}{3}\left(2\,\frac{d\omega}{dy}\bigg|_{n-2} - \frac{d\omega}{dy}\bigg|_{n-1} + 2\,\frac{d\omega}{dy}\bigg|_{n}\right), \tag{2.21}$$

Simpson's formula, which is used as a check,

$$\omega_{n+1}^{(2)} = \omega_{n-1} + \frac{\Delta y}{3}\left(\frac{d\omega}{dy}\bigg|_{n-1} + 4\frac{d\omega}{dy}\bigg|_{n} + \frac{d\omega}{dy}\bigg|_{n+1}\right) \tag{2.22}$$

and the error term

$$\text{error} = \frac{1}{29}\left(\omega_{n+1}^{(1)} - \omega_{n+1}^{(2)}\right). \tag{2.23}$$

Milne's formula passes a curve through four points and estimates the value of the function at the fifth point by an extrapolation of the curve, (see figure 7). When the value of $\omega_{n+1}^{(1)}$ is calculated with the aid of formula (2.21), the given differential equation is used to find the slope $\left.\frac{d\omega}{dy}\right|_{n+1}$. The value of $\left.\frac{d\omega}{dy}\right|_{n+1}$ obtained from the above procedure is used in Simpson's formula, (2.22), to make a second approximation for $\omega_{n+1}^{(2)}$.

When exact values are used in (2.21) and (2.22), formula (2.21) gives an ω_{n+1} which is too small while (2.22) gives an ω_{n+1} which is too large. A consideration of the terms which are neglected in the extrapolation formulae (2.21) and (2.22) shows that the maximum deviation from the true curve is given by the formula (2.23). Thus, formula (2.22) provides a check on the numerical calculations for (2.21), and (2.23) gives a means of determining the accuracy of the approximation method.

Before Milne's Method can be applied, the values of the function and its first derivatives at four points must be obtained. The value of the function at the second point is found with the aid of Euler's averaging method. The formula used in this method is

$$\omega_1 = \omega_o + \frac{\Delta y}{2}\left(\left.\frac{d\omega}{dy}\right|_o + \left.\frac{d\omega}{dy}\right|_1\right) . \qquad (2.24)$$

The value of the function at the third and fourth points is found by applying Simpson's formula, (2.22). The numerical procedure is explained by showing the steps involved in solving equations (2.17) and (2.18) when the initial conditions are $\omega_{(o)} = 0$ and $a_{(o)} = \frac{1}{2}$.

The solutions of equations (2.15) and (2.16) for $d\omega/dy$ and da/dy are

$$d\omega/dy = \frac{\left\{\left[0.6928\ a^3 \sin\omega \cos(\omega + \pi/6) - a(1 - 0.2a^2)\sin 2\omega + 0.866\ y^2 \sin(\omega + \pi/6)\right]\right\}}{ya\left[1 - 0.6a^2 + 0.4a^2 \sin^3(\omega + \pi/6)\right]} \tag{2.25}$$

$$-\, da/dy = \frac{2a(1 - 0.2a^2)\sin^2\omega + 0.866\ y^2 \cos(\omega + \pi/6)}{y\left[1 - 0.6a^2 + 0.4a^2 \sin^2(\omega + \pi/6)\right]} \tag{2.26}$$

For the initial starting value of $\omega = 0$, formulae (2.25) and (2.26) yield indeterminate equations for $\frac{d\omega}{dy}$ and $\frac{da}{dy}$. These equations are of the form 0/0. The evaluation of (2.25) and (2.26) at $\omega = 0$ can be accomplished by expanding $\sin\omega$ and $\sin^2\omega$ about $\omega = 0$ in terms of y. These expansions are

$$\sin\omega = y \left.\frac{d\omega}{dy}\right|_o + \frac{y^2}{2!} \left.\frac{d^2\omega}{dy^2}\right|_o + \ldots$$

$$\sin 2\omega = 2y \left.\frac{d\omega}{dy}\right|_o + y^2 \left.\frac{d^2\omega}{dy^2}\right|_o + \ldots$$

With the substitution of these expansions into (2.25) and (2.26), the limits of the expressions for the derivatives as $y \to 0$ are

$$\lim_{y \to 0} \frac{d\omega}{dy} = -2 \left.\frac{d\omega}{dy}\right|_o , \quad \lim_{y \to 0} \frac{da}{dy} = 0 .$$

This procedure yields starting values for $d\omega/dy$ and da/dy of

$$\frac{d\omega}{dy} = 0 \quad , \quad \frac{da}{dy} = 0 . \tag{2.27}$$

For an increment in y, steps of 0.05 are chosen. The initial increment in y should be small because of the irregular behavior of ω at the origin. Also, the checking methods do not apply until the values for a, ω, $d\omega/dy$, and da/dy are calculated for four values of y.

First step: Calculation for the point $y = 0.05$.

The values $\omega = 0$ and $a = 0.5$ in the row $y = 0$ in Table II are the prescribed initial conditions. First approximations for ω and a at $y = 0.05$ are assumed to be $\omega = 0$ and $a = 0.5$. First approximations for $d\omega/dy$ and da/dy at $y = 0.05$ are obtained by using these values of ω and a in formulae (2.25) and (2.26). The newly calculated $d\omega/dy$ and da/dy are substituted in formulae (2.24) to obtain second approximations of ω and a. These values of ω and a can be substituted in formulae (2.25) and (2.26) to calculate second approximations for $d\omega/dy$ and da/dy. With these values of the derivatives, third approximations for ω and a are obtained from the expression (2.24).

This process of approximation is repeated until the values for $d\omega/dy$ and da/dy remain stationary in which case the ω and a given by (2.24) remain constant. These stationary values of ω, "a" and their derivatives are given in the row $y = 0.05$ of Table II.

In the above procedure, the values of ω oscillate about the stationary ω with a decreasing amplitude. With the use of slight alterations in the value of ω towards the center of the oscillation, the number of calculations can be reduced. The magnitude of these alterations can be estimated after the first two calculations.

Second Step: Calculations for the points $y = 0.10$ and $y = 0.15$.

Approximate values of ω and a at $y = 0.10$ are obtained with the use of Simpson's formula, (2.27). In Simpson's formula, however, the value of the derivative at the point to be calculated must be known. Therefore, an estimation of this derivative is made by a linear extrapolation of the two preceding derivatives. The approximate values of ω and a calculated by Simpson's formula are substituted into expressions (2.25) and (2.26) to obtain new estimates of the derivatives. These values of the derivatives are used to make new approximations for ω and "a" with Simpson's formula. Two calculations are usually sufficient to obtain the desired accuracy for the value of the functions and their derivatives. Exactly this same procedure is used to make the calculations for ω and "a" at $y = 0.15$. The results of these calculations are shown in Table II.

Third Step: Calculations for further values of y.

Once four values of ω, a, and their derivatives are obtained, the regular numerical integration procedure is adopted. Formula (2.21) gives the values $\omega_{n+1}^{(1)}$ and $a_{n+1}^{(1)}$ for $y = 0.2$. These values are substituted into formulae (2.25) and (2.26) to calculate $d\omega/dy_{\,n+1}$ and $da/dy_{\,n+1}$. These values of ω, a, and their derivatives are entered in the row $y = 0.2$.

All the quantities which are necessary to use formula (2.22) are obtained. The second calculations for $\omega_{n+1}^{(2)}$ and $a_{n+1}^{(2)}$ are made with Simpson's formula. An estimate of the error is made with expression (2.23) where mistakes in the numerical work or too large a choice of the increment produce a maximum error which is too large.

The accuracy of this method is such that larger increments can be used once enough starting values are calculated. Thus, when values for ω, a, and their derivatives are calculated up to $y = 0.3$, enough information is available to use Milne's Method with increments of

$y = 0.1$. Table III is constructed to show the calculations for increments of $y = 0.1$.

Fourth Step: Calculation of the outer boundary of the plate.

The boundary condition for ω at the outer edge of the plate is $\omega = \pi/3 = 1.0472$. In this example the value of ω at $y = 1.1$ is greater that $\pi/3$.

An extrapolation procedure is used to obtain the outer boundary. The equation for this extrapolation is based on Taylor's expansion expressed in terms of differences.

$$\omega_b = \omega_k + n \Delta \omega_k + \frac{n(n-1)}{2} \Delta^2 \omega_k$$

where (2.28)

$$\Delta\omega_k = \omega_{k+1} - \omega_k \;, \; \Delta^2\omega_k = \omega_{k+2} - 2\omega_{k+1} + \omega_k .$$

The value of "n" is $2 \leq n \leq 3$.

A pictorial representation is given in figure 7b. All the quantities of the above equation are known except n. The determination of n gives sufficient information to obtain the outer boundary of the plate. Reorganization of formula (2.28) gives this quadratic equation for n.

$$0 = n^2 (\omega_{k+2} - 2\omega_{k+1} + \omega_k) + n (- \omega_{k+2} + 4\omega_{k+1} - 3\omega_k) - 2 (\omega_b - \omega_k) \tag{2.29}$$

The quantity ω_b is 1.0472. In this example, the solution of (2.29) is $n = 2.558$. The formula for the outer boundary of the plate is

$$Y = y_b = y_k + n \Delta y. \tag{2.30}$$

In this example $y_k = 0.8$, $\Delta y = 0.1$, and $n = 2.558$; therefore, Y is $Y = 1.0558$. Equation (2.28) is reorganized and written in terms of a.

$$a_b = a_k + \frac{h^2}{2} (a_{k+2} - 2a_{k+1} + a_k) + \frac{h}{2} (- a_{k+2} + 4a_k - 3a_k) \qquad (2.31)$$

In (2.31) all the quantities are known except a_b; thus the value of "a" at the outer boundary of the plate can be found with formula (2.31).

Table III gives the numerical results of this example while figure 8 is the graphical presentation. With the use of (2.13), (2.14), and (2.19), equations (2.1) are expressed as

$$\alpha_r = - \frac{2}{3} \frac{a}{Y^2} \frac{pb^2}{D} \sin (\omega - \pi/6)$$

$$\alpha_\theta = \frac{2}{3} \frac{a}{Y^2} \frac{pb^2}{D} \sin (\omega + \pi/6) , \qquad (2.32)$$

and (2.4) as

$$M_r = \frac{(1 - 0.2a^2)a}{\sqrt{3}\, Y^2} pb^2 \cos (\omega + \pi/6),$$

$$M_\theta = \frac{(1 - 0.2a^2)a}{\sqrt{3}\, Y^2} pb^2 \cos (\omega - \pi/6) . \qquad (2.33)$$

From the results of the solution of (2.32) for α_θ, the deflection of the mean surface of the plate, $w(r)$, is calculated by a numerical integration of (2.34), the transformed equation I of Table I.

$$w_{(r)} = \int_r^b r\alpha_\theta \, dr \quad \text{where} \quad w_{(b)} = 0. \qquad (2.34)$$

The actual calculations for α_θ and $w(r)$ are performed in Chapter 6 where the different approximate methods are compared.

TABLE II

y	ω'	ω	- a'	a
0	0	0	0	.5
0.05	0.0248	0.0006	0.0428	0.4989
0.10	0.0501	0.0025	0.0854	0.4957
0.15	0.0766	0.0056	0.1276	0.4904
0.20	0.1045	0.0102	0.1692	0.4830
0.25	0.1353	0.0161	0.2099	0.4735
0.30	0.1672	0.0238	0.2497	0.4620

TABLE III

y	ω'	ω	- a'	a
0.0	0	0	0	0.5
0.1	0.0501	0.0025	0.0854	0.4957
0.2	0.1045	0.0102	0.1692	0.4830
0.3	0.1672	0.0238	0.2497	0.4620
0.4	0.2484	0.0452	0.3257	0.4331
0.5	0.3678	0.0743	0.3952	0.3970
0.6	0.5377	0.1197	0.4587	0.3545
0.7	0.8038	0.1844	0.5103	0.3055
0.8	1.3603	0.2859	0.5412	0.2528
0.9	2.4360	0.4733	0.5207	0.1984
0.10	4.5692	0.8023	0.3680	0.1517
1.0558		1.0472		0.1290

2.4 Evaluation of the Method of Sokolovsky.

The method of Sokolovsky gives a procedure for obtaining a numerical solution of equations II, III, and IV of Table I. This solution is obtained by the simultaneous integration of the two non-linear first order differential equations (2.8) and (2.9). The actual numerical integration of these equations is greatly simplified by assuming expression (2.11) for a stress-strain relation. Expression (2.11) is a parabolic approximation for the actual stress-strain relationship. The accuracy of the results given in figure 8 and 9 depends upon the accuracy of this assumption. Figure 9 is a graph showing the dependence of ω and "a" upon the radius of the plate, r, when the initial values of a = 0.25, a = 0.50 and a = 1.00 are used to calculate the example of section 2.2.

A check on the accuracy of each step of the numerical integration procedure is given by formula (2.23). Though the error of each step is small, the error of the whole procedure is cumulative. Equations (2.32) and (2.33) show that the cumulative error in determining Y alters the value of the moments and curvatures throughout the plate. When the same size increment is used, the behavior of the integration process is such that the error is greater for smaller initial "a". Thus, this method gives better results for plates which are loaded near to the limiting strain condition, or for initial "a" approximately one.

The method of this chapter is an inverse method. The variable "a", which is a measure of the strain condition of the plate, is given an initial value, and the boundary condition (2.19) determines the parameter γ^2 equal to $\lambda^{1/2}$ hpb^2/D. The parameter Υ determines the physical properties of the plate and the permissible load. Any one of the factors in Υ can be adjusted to give the correct boundary value of Υ. But if a plate of a given material and specific dimensions has a definite pressure applied, Υ is determined. Thus, a series of calculations are necessary to adjust the initial value of "a" so that the correct value of Υ occurs at the boundary of the plate.

The calculation of an actual example, for any given initial conditions, can be performed within a day. This calculation gives ω and "a" as a function of y in the form of a table or a graph. Thus, the bending moments, curvatures, and deflections of the plate, given by (2.32), (2.33), and (2.34), cannot be expressed analytically. A complete description of the physical behavior of the plate can be found to any desired accuracy when sufficiently small increments are used. But, this method is too lengthy for quick estimates of the moments, curvatures, or displacements for plates with prescribed dimensions. In the chapters which follow several approximate methods are described which can provide these quick estimates. The last chapter of this thesis presents a comparison of the results of these various methods.

CHAPTER III

THE ITERATION METHOD

3.1 The Mathematical Development of the Method.

The paper by Ilyushin (9) develops an approximate method for the solution of problems with plastic stress-strain relations. This method leads to differential equations which can be solved by an iteration procedure, and is used here to obtain an approximate solution for the problem of the bending of a circular plate.

A second order ordinary differential equation is obtained by the substitution of formulae II and IV, into III and V of Table I.

$$r \frac{d^2\alpha_\theta}{dr^2} + 3 \frac{d\alpha_\theta}{dr} = - \frac{\int_0^r rp(r)dr}{D} + B \left(r \frac{d^2\alpha_\theta}{dr^2} + 3 \frac{d\alpha_\theta}{dr}\right)$$

$$+ \frac{dB}{dr} \left(r \frac{d\alpha_\theta}{dr} + \frac{3}{2} \alpha_\theta\right) \tag{3.1}$$

where

$$B = \sum_{n=1}^{\infty} \frac{3\lambda_n h^{2n}}{2n+3} \left[r^2 \left(\frac{d\alpha_\theta}{dr}\right)^2 + 3r\alpha_\theta \frac{d\alpha_\theta}{dr} + 3 \alpha_\theta^2 \right]^n . \tag{3.2}$$

The iteration method of Ilyushin is now applied to equation (3.1).

Equation (3.1) is reorganized into

$$\frac{d\left[\frac{1}{r} \frac{d(r^2\alpha_\theta)}{dr}\right]}{dr} = - \left\{ \frac{Q_r}{D} - \left(\frac{d\left[B \left(r \frac{d\alpha_\theta}{dr}\right) + \frac{3}{2} \alpha_\theta\right]}{dr} + \frac{B}{2} \frac{d\alpha_\theta}{dr} \right) \right\}, \tag{3.3}$$

where

$$Q_r = \frac{1}{r} \int_0^r rp(r)dr. \tag{1.5}$$

The terms on the right hand side of (3.3) are separated into

$$V_e = Q_r, \tag{3.4}$$

and

$$V_p = -D\left\{\frac{d}{dr}\left[B\left(r\frac{d\alpha_\theta}{dr} + \frac{3}{2}\alpha_\theta\right)\right] + \frac{B}{2}\frac{d\alpha_\theta}{dr}\right\}. \tag{3.6}$$

The expression on the left hand side of (3.3) is encountered in the solution for the elastic bending of a circular plate. When the stress-strain relation is linear, the quantity B, given by (3.2) is zero. The linear formula for a circular plate is

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r^2\alpha_\theta)\right] = -\frac{V_e}{D}.$$

From (3.4) and (3.5), equation (3.3) can be rewritten as

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r^2\alpha_\theta)\right] = -\frac{1}{D}(V_e + V_p). \tag{3.6}$$

The quantity V_p, which is interpreted as a pseudo-shearing force due to the non-linear stress strain relation, can be evaluated in terms of an estimated α_θ as a function of r. An approximate solution for α_θ is then found by solving (3.6) with the shear V_e and the estimated pseudo-shear V_p. The approximate solution of this section will not be found directly from (3.3), but a convenient form suitable for an iteration procedure is derived from (3.3) by using a Green's Function. The example of a simply supported plate is used to demonstrate this method.

The boundary conditions for a simply supported plate are:

a). At the center of the plate

$$r = 0,\ M_r = M_\theta,\ \alpha_\theta \text{ is finite} \tag{3.7}$$

b). At the outer edge of the plate

$$r = b, \ M_r = 0, \ r\frac{d\alpha_\theta}{dr} + \frac{3}{2}\alpha_\theta = 0. \tag{3.8}$$

Expressions (3.7) and (3.8) are derived by substituting the given values of the moments into formulae III, Table I.

Equation (3.3) will be expressed in an integral form from which α_θ can be found by integration for any arbitrary shear force. For this integral representation, a solution, satisfying the boundary conditions (3.7) and (3.8), is required for

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d}{dr}(r^2\alpha_\theta)\right] = -\delta(\xi) \tag{3.9}$$

where

$\delta(\xi) = 0$, for $0 \leq r < \xi$, $\xi < r \leq b$,

and at $r = \xi$, $\delta(\xi)$ increases $\int_0^r d\left[\frac{1}{r}\frac{d}{dr}(r^2\alpha_\theta)\right]$ one unit. The first integration of (3.9) yields

$$\frac{1}{r}\frac{d}{dr}(r^2\alpha_\theta) = C_1 \qquad 0 \leq r < \xi$$

$$\tag{3.10}$$

$$\frac{1}{r}\frac{d}{dr}(r^2\alpha_\theta) = C_1 - 1 \qquad \xi < r \leq b.$$

The second integration of (3.9) yields

$$\alpha_\theta = \frac{C_1}{2} + \frac{C_2}{r^2} \qquad 0 \leq r < \xi \tag{3.11}$$

$$\alpha_\theta = \frac{(C_1 - 1)}{2} + \frac{C_3}{r^2} \qquad \xi < r \leq b. \tag{3.12}$$

The boundary conditions (3.7) and (3.8) and the condition that α_θ is continuous at $r = \xi$ determine the three constants C_1, C_2, and C_3.

These constants are

$$C_1 = \frac{1}{3}\ 3 + (\frac{\xi}{b})^2,\ C_2 = 0,\ C_3 = \frac{\xi^2}{2},$$

which upon substitution into (3.11) and (3.12) determine the solution for α_θ as

$$\alpha_\theta = \frac{(1)}{6}\left[3 + (\frac{\xi}{b})^2\right] \qquad 0 \le r < \xi$$

$$\alpha_\theta = \frac{(1)}{6}\left[(\frac{\xi}{b})^2 + 3\ (\frac{\xi}{r})^2\right] \qquad \xi < r \le b \tag{3.13}$$

The particular solution (3.13) is used to obtain the effect of an arbitrarily distributed shearing force, $V(\xi)$. The magnitude of the incremental shearing force acting on the area $d\xi$ at $r = \xi$ is

$$\delta\ (\xi)\ \frac{V(\xi)}{D}\ d\xi\ .$$

This shearing force produces an incremental change in α_θ of

$$d\ \alpha_\theta(r) = \frac{V(\xi)\ d\xi}{6D}\ \left[3 + (\frac{\xi}{b})^2\right] \qquad 0 \le r < \xi$$

$$d\ \alpha_\theta(r) = \frac{V(\xi)\ d\xi}{6D}\ \left[(\frac{\xi}{b})^2 + 3\ (\frac{\xi}{r})^2\right] \quad \xi < r \le b.$$

Due to the linearity of equation (3.9), the effects of the incremental shearing forces can be integrated to form the total increase in α_θ caused by the distributed shearing force $V(\xi)$. This increase is

$$\alpha_{\theta(r)} = \int_0^b \frac{1}{6}\ (\frac{\xi}{b})^2\ \frac{V(\xi)}{D}\ d\xi + \int_0^r \frac{1}{2}\ (\frac{\xi}{r})^2\ \frac{V(\xi)}{r}\ d\xi$$

$$+ \int_r^b \frac{1}{2}\ \frac{V(\xi)}{D}\ d\xi\ . \tag{3.14}$$

The integral form of equation (3.3) is obtained when the expressions (3.4) and (3.5) are substituted into (3.14) for the arbitrary shearing force $V(\xi)$.

$$V(\xi) = V_e + V_p.$$

The result of the above procedure, after reorganization and several integrations by parts to eliminate the derivatives of B, is

$$\alpha_{\theta(r)} = \frac{1}{6}\int_0^b \frac{Q_r}{D}\left(\frac{\xi}{b}\right)^2 d\xi + \frac{1}{2}\int_0^r \frac{Q_r}{D}\left(\frac{\xi}{r}\right)^2 d\xi + \frac{1}{2}\int_r^b \frac{Q_r}{D}\, d\xi$$

$$+ \frac{1}{2}\int_0^b \frac{\xi}{b^2}\left(\alpha_\theta + \frac{\xi}{2}\frac{d\alpha_\theta}{d\xi}\right) B\,d\xi + \frac{3}{2}\int_0^r \frac{\xi}{r^2}\left(\alpha_\theta + \frac{\xi}{2}\frac{d\alpha_\theta}{d\xi}\right) B\,d\xi \qquad (3.15)$$

$$- \frac{1}{4}\int_r^b \frac{d\alpha_\theta}{d\xi} B\,d\xi .$$

The form of this equation ensures the satisfaction of the boundary conditions for the simply supported plate. Boundary conditions (3.7) and (3.8) are applied in the formulation of the Green's Function which determines α_θ for arbitrarily distributed shearing forces. For a plate subjected to different boundary conditions, equations (3.7) and (3.8) are replaced by new equations. The subsequent formulae are altered, but the procedure remains the same.

Expression (3.15) is the desired integral form of (3.3) for the application of the iteration method. The explanation of this iteration process is explained in the next section by solving the example problem.

The integral form of equation (3.3) is obtained when the expressions (3.4) and (3.5) are substituted into (3.14) for the arbitrary shearing force $V(\xi)$.

$$V(\xi) = V_e + V_p.$$

The result of the above procedure, after reorganization and several integrations by parts to eliminate the derivatives of B, is

$$\alpha_{\theta(r)} = \frac{1}{6}\int_0^b \frac{Q_r}{D}\left(\frac{\xi}{b}\right)^2 d\xi + \frac{1}{2}\int_0^r \frac{Q_r}{D}\left(\frac{\xi}{r}\right)^2 d\xi + \frac{1}{2}\int_r^b \frac{Q_r}{D}\, d\xi$$
$$+ \frac{1}{2}\int_0^b \frac{\xi}{b^2}\left(\alpha_\theta + \frac{\xi}{2}\frac{d\alpha_\theta}{d\xi}\right) Bd\xi + \frac{3}{2}\int_0^r \frac{\xi}{r^2}\left(\alpha_\theta + \frac{\xi}{2}\frac{d\alpha_\theta}{d\xi}\right) Bd\xi \qquad (3.15)$$
$$- \frac{1}{4}\int_r^b \frac{d\alpha_\theta}{d\xi}\, Bd\xi\,.$$

The form of this equation ensures the satisfaction of the boundary conditions for the simply supported plate. Boundary conditions (3.7) and (3.8) are applied in the formulation of the Green's Function which determines α_θ for arbitrarily distributed shearing forces. For a plate subjected to different boundary conditions, equations (3.7) and (3.8) are replaced by new equations. The subsequent formulae are altered, but the procedure remains the same.

Expression (3.15) is the desired integral form of (3.3) for the application of the iteration method. The explanation of this iteration process is explained in the next section by solving the example problem.

3.2 The Numerical Example.

The example problem of section 2.2 is used to illustrate the iteration process. In this example the load is a constant.

$$p(r) = p$$

$$Q_r = 1/r \int_0^r prdr = pr/2 \tag{3.16}$$

A parabolic stress-strain relation is assumed which determines the quantity B from (3.2) as

$$B = 0.6\,\lambda h^2 \left[3\,\alpha_\theta^{\,2} + 3\,\alpha_\theta \left(r\,\frac{d\alpha_\theta}{dr}\right) + \left(r\,\frac{d\alpha_\theta}{dr}\right)^2\right] \tag{3.17}$$

With the substitution of (3.16) into equation (3.15), α_θ is

$$\alpha_{\theta(r)} = \frac{pb^2}{48D}\left[7 - \left(\frac{r}{b}\right)^2\right] + \frac{1}{2}\int_0^b \frac{\xi}{b^2}\left(\alpha_\theta + \frac{d\alpha_\theta}{2}\right) Bd\xi$$

$$+ \frac{3}{2}\int_0^r \frac{\xi}{r^2}\left(\alpha_\theta + \frac{\xi}{2}\,\frac{d\alpha_\theta}{d}\right) Bd\xi - \frac{1}{4}\int_r^b \frac{d\alpha_\theta}{d}\, Bd\xi \,. \tag{3.18}$$

The iteration procedure is now applied to equation (3.18).

a). The elastic solution.

The first step of the iteration process is to assume that $\lambda = 0$. This is equivalent to assuming that B is zero or that the pseudo-shearing force V_p, equation (3.5), is not present. The α_θ calculated from this assumption is given by a plate with an elastic stress-strain relation as

$$\alpha_\theta^{(e)} = \frac{pb^2}{48D}\left[7 - 3\left(\frac{r}{b}\right)^2\right], \tag{3.19}$$

which is obtained by evaluating (3.18) when B = 0.

b). First approximation for α_θ.

The first approximation for α_θ is found by integrating (3.18) where (3.18) is evaluated by substituting (3.17) for B, and $\alpha_\theta^{(e)}$ from formula (3.19) for α_θ. We find

$$\alpha_{\theta(r)}^{(1)} = \alpha_\theta^{e} + \frac{pb^2}{48D} Y^4 f_1 \tag{3.20}$$

$$Y^2 = \lambda^{1/2}\ hpb^2/D$$

$$f_1 = 0.2327 - 0.2871\left(\frac{r}{b}\right)^2 + 0.1764\left(\frac{r}{b}\right)^4 - 0.04189\left(\frac{r}{b}\right)^6.$$

The dimensionless parameter Y is the same parameter encountered in section (2.2). This step is equivalent to approximating the pseudo-shearing force V_p, formula (3.5), in terms of the elastic effects on α_θ. The apparent shearing force V_p, due to the plastic behavior of the material, increases α_θ.

c). The second approximation for α_θ.

The value fc $\alpha_\theta^{(1)}$ obtained in expression (3.20) can be used to evaluate B and α_θ in (3.17) and (3.18) as functions of r. Then expression (3.18) is integrated for the second approximation for

$$\alpha_\theta^{(2)} = \alpha_\theta^{(e)} + \frac{pb^2}{48D}\left[y^4 f_1 + y^8 f_2 + 0_{12}(Y)\right] \tag{3.21}$$

$$f_2 = 0.0455 - 0.0931\left(\frac{r}{b}\right)^2 + 0.113\left(\frac{r}{b}\right)^4 - 0.0793\left(\frac{r}{b}\right)^6 + 0.0302\left(\frac{r}{b}\right)^8 - 0.00454\left(\frac{r}{b}\right)^{10}.$$

In this approximation another estimation for the apparent shearing force V_p is made by using a rectified α_θ. The increase in the estimated $\alpha_\theta^{(1)}$ over $\alpha_\theta^{(e)}$ gives a greater value for $\alpha_\theta^{(2)}$ than $\alpha_\theta^{(1)}$. The term

$O_{12}(Y)$ denotes terms containing the twelfth and higher powers of Y. These terms are altered on further approximations for α_θ; yet they are not needed for the calculation of the twelfth power terms in Y in the next approximation.

d). The n^{th} approximation for α_θ.

For the n^{th} approximation of α_θ, $\alpha_\theta^{(n-1)}$ is used to evaluate the integral (3.18). The form of this solution is

$$\alpha_\theta^{(n)} = \alpha_\theta^{(e)} + \frac{pb^2}{48D}\left[\sum_{k=1}^{n} Y^{4k} f_k + O_{4(n+1)}(Y)\right] . \tag{3.22}$$

The term $Y^{4n} f_n$ is the additional correction for α_θ. This term is not altered in further approximations. The terms of higher order than 4n in the parameter Y can be neglected and do not effect the next approximation for the term containing Y^{4n+4}. The elimination of the terms $O_{yn+4}(Y)$ in each approximation greatly decreases the numerical work.

e). The check.

For each approximation a new function, f_n, is obtained. This function is the only term which has the parameter Y to the 4n power. For the computations to be correct, all the functions, f_k, must satisfy the boundary conditions. After each approximation the newly acquired function f_n should be substituted into expression (3.8) to check the computations.

The functions f_1 and f_2 are evaluated in Table IV for various ratios of r/b. These values can be used to calculate $\alpha_\theta^{(2)}$ for various loads and values of the parameter Y.

TABLE IV

Evaluation of functions appearing in the Iteration Method. (Formula 3.21)

r/b	$7 - 3(r/b)^2$	$f_1\left[(r/b)^2\right]$	$f_2\left[(r/b)^2\right]$
0	7.00	0.233	0.0455
0.1	6.97	0.230	0.0446
0.2	6.88	0.222	0.0420
0.3	6.73	0.208	0.0380
0.4	6.52	0.191	0.0332
0.5	6.25	0.171	0.0282
0.6	5.92	0.150	0.0234
0.7	5.53	0.129	0.0193
0.8	5.08	0.110	0.0160
0.9	4.57	0.0936	0.0135
1.0	4.00	0.0801	0.0118

3.3 Evaluation of the Method.

The iteration method provides a solution for the plastic bending of a circular plate in terms of the dimensionless parameter Y and functions of r. In section 2.2 values of Y are calculated for various degrees of plastic yielding. For a condition of complete plastic yielding at the center of the plate $Y = 1.368$. Therefore, the range of Y is $0 \leq Y \leq 1.386$. The accuracy of this method for a given number of approximations depends upon the convergence of f_k and the value of Y.

With the use of Table IV, an estimate of the maximum correction for α_θ between a first and second approximation is about 7 percent. This estimate is obtained from

$$\text{percent correction} = \frac{Y^8 f_2}{7 - 3(\frac{r}{b})^2 + Y^4 f_1} \; 100 \; . \tag{3.23}$$

The error involved in the two term approximation should be less than this value.

Each iteration increases the estimated value of α_θ, which makes the series of the approximate pseudo shearing force $V_p^{(n)}$ a monotonously increasing series. Physical considerations make it plausible that the correct V_p is the limit of this series or that the iteration method is convergent. Panferov (5) treats the problem of convergence for this method mathematically.

The exact solution for α_θ in terms of the infinite series obtained by the iteration procedure is

$$\alpha_\theta = \frac{pb^2}{48D}\left[7 - 3(\frac{r}{b})^2 + \sum_{k=1}^{\infty} Y^{4k} f_k\right] . \tag{3.24}$$

The deviation from the n^{th} approximation for the $\alpha_\theta^{(n)}$ given by (3.22) and the exact α_θ is

$$\alpha_\theta - \alpha_\theta^{(n)} = \frac{pb^2}{48D} \sum_{k=n+1}^{\infty} Y^{4k} f_k. \tag{3.25}$$

Expression (3.25) shows that the sequence $\alpha_\theta - \alpha_\theta^{(n)}$ converges more slowly the greater the parameter Y. This result follows from the consideration that the greater the value of the parameter Y, which indicates the extent of plastic straining, the greater is the difference $\alpha_\theta - \alpha_\theta^{(e)}$. Thus, a greater number of approximations are needed to correct the larger initial error for large values of the parameter Y.

When the desired accuracy must be greater than that given by a two term approximation, Sokolovsky's Method is preferable. The numerical work for higher term approximations of the iteration method is excessive. This method has advantages over Sokolovsky's Method because the approximations involve only polynomials in r which can be integrated or differentiated to determine the deflection and α_r. The mathematical approach of the iteration method is not as direct as Sokolovsky's Method, but the actual calculations are shorter for any problem with a given load distribution and boundary conditions when the two term approximation is used. Only one set of calculations is needed for any outer radius of the plate or magnitude of the parameter determining the load distribution.

CHAPTER IV

THE APPLICATION OF POTENTIAL ENERGY

4.1 Introduction to the Minimum Potential Energy Method.

When energy methods or variational principles are used, the variational principle should be thought of as the basic law governing the behavior of the system. The behavior of the system is determined by a definite variational procedure upon the energy or equivalent quantity of the complete system. This method of approach differs from the procedure of the previous chapters where the behavior of every element is analyzed. Both methods are equivalent, but the assumed fundamental concepts are different.

The variational principle used in this chapter is the principle of minimum potential energy. Ilyushin (9) and Phillips (10) give a general development of the principle of minimum potential energy. Greenberg (16) treats the subject of variational methods for the theory of plastic flow and the theory of plastic deformations. His report gives a thorough development and includes the original references for the various variational principles.

The next section directly approaches the problem of the deformation of a circular plate using the minimum potential energy method. The problem is expressed in terms of the moments, curvatures, and deflections of the plate.

4.2 Formulation of the Minimum Potential Energy Method for the Bending of Circular Plates.

For the moment-curvature relations IV of Table I, the strain energy per unit of a plate is expressed in terms of A as

$$W = \int_{0}^{A^2} \frac{D(1 - B)}{2} \, dA^2 \, . \tag{4.1}$$

The potential energy for the plate is

$$\Phi = 2\pi \int_{0}^{b} (W - pw) \, rdr. \tag{4.2}$$

For the variational principle of this section the following definitions are necessary:

1). Admissible displacements are displacements which agree with the prescribed boundary conditions.

2). Admissible curvatures are curvatures which satisfy equations I and II of Table I throughout the plate, where the w in equation I of Table I is an admissible displacement.

3). Admissible moments are moments derived from admissible curvatures by equations IV of Table I. The load p is a given quantity which is not varied. From equations (2.2) for A and (4.1) for W, we find that Φ can be expressed in terms of only the admissible curvatures and displacements.

The principle of minimum potential energy states that the admissible curvatures and displacements which render Φ a minimum yield admissible moments that satisfy the equilibrium condition III of Table I. Thus, these curvatures are a solution for the bending of a circular plate which satisfy equations I, II, III, and IV of Table I. The proof of this principle follows:

A necessary condition for Φ to be a minimum is that the first variation of Φ is zero for all admissible curvatures and displacements. Hence,

$$\delta\bar{\Phi} = 2\pi \int_0^b \left[\frac{D(1-B)}{2} \frac{\partial A^2}{\partial \alpha_r} \delta\alpha_r + \frac{D(1-B)}{2} \frac{\partial A^2}{\partial \alpha_\theta} \delta\alpha_\theta - p\delta w \right] rdr = 0 \tag{4.3}$$

From the moment-curvature relations III of Table I and formula (2.2), the moments are expressed in terms of the curvatures as

$$M_r = \frac{D(1-B)}{2} \frac{\partial A^2}{\partial \alpha_r} \; ; \; M_\theta = \frac{D(1-B)}{2} \frac{\partial A^2}{\partial \alpha_\theta} . \tag{4.4}$$

The substitution of equations (4.4) into (4.3) yields

$$\int_0^b (M_r \delta\alpha_r + M_\theta \delta\alpha_\theta - p\delta w) \, rdr = 0 \tag{4.5}$$

The variations of the displacements and curvatures must satisfy equations I and II of Table I:

$$\begin{aligned} \delta\alpha_\theta &= -\frac{1}{r} \frac{d(\delta w)}{dr} , \\ \delta\alpha_r &= \frac{d(r\delta\alpha_\theta)}{dr} . \end{aligned} \tag{4.6}$$

With the substitution of the following identities

$$\begin{aligned} \int_0^b rM_r \frac{d(r\delta\alpha_\theta)}{dr} dr &= r^2 M_r \, \delta\alpha_\theta \Big|_0^b - \int_0^b \frac{d(rM_r)}{dr} \delta\alpha_\theta \, rdr \\ &= 0 - \int_0^b \frac{d(rM_r)}{dr} \delta\alpha_\theta \, rdr \end{aligned} \tag{4.7}$$

$$\int_0^b \delta w \; prdr = \delta w \int_0^r rpdr \Big|_0^b + \int_0^b \left(\int_0^r rpdr\right) \left[-\frac{1}{r}\frac{d(\delta w)}{dr}\right] rdr$$
$$= 0 + \int_0^b \left(\int_0^r rpdr\right) \left[-\frac{1}{r}\frac{d(\delta w)}{dr}\right] rdr \qquad (4.7)$$

and equations (4.6) into (4.5), we find

$$\int_0^b \left[\frac{d(rM_r)}{dr} - M_\theta + \int_0^r rpdr\right] \delta\alpha_\theta \; rdr = 0 \qquad (4.8)$$

When expression (4.8) is satisfied for all possible admissible variations of α_θ, the bracketed term must be identically zero. This bracketed term is the equilibrium equation IV of Table I; thus, the principle of minimum potential energy is proved.

4.3 Approximate Methods.

Two approximate methods are available for the solution of equations (4.2) and (4.8). These methods place restrictions upon arbitrary parameters in an assumed form of a solution. The explanation of these two methods follows:

The first method, the Rayleigh-Ritz Method, assumes an approximate form of the deflection w with the adjustable parameters C_i,

$$\bar{w} = \sum_{i=1}^{n} C_i \; w_{i(r)}, \qquad (4.9)$$

where each w_i satisfies the boundary condition imposed upon the plate. An approximation for the potential energy of the plate is found by using expression (4.9) to evaluate the integral (4.2).

$$I = \int_0^b \left[W(\bar{w}) - p(r)\bar{w}\right] rdr \qquad (4.10)$$

For the calculation of $W(\bar{w})$, the curvatures are obtained with equations I and II of Table I. The parameters C_i are varied to give a stationary value of the integral (4.10) by setting its partial derivative with respect to each of the parameters equal to zero.

$$\partial I/\partial C_i = 0 \qquad i = 1, \ldots, n \tag{4.11}$$

Equations (4.11) are n equations which determine the n unknown parameters C_i. The extremum of I is found for the restricted set of admissible displacements $\bar{w}$. Other functions not contained in the above set may yield a lower potential energy; therefore, this method only gives an approximation for the strain condition of the plate.

The second method, Galerkin's Method, uses equation (4.8) to make an approximation for α_θ. This method assumes an approximate α_θ with the parameters C_i as

$$\bar{\alpha}_\theta = \sum_{i=1}^{n} C_i \, \alpha_{\theta i}(r). \tag{4.12}$$

Each $\alpha_{\theta i}$ should satisfy the boundary conditions of the plate. The C_i are determined by evaluating

$$\int_0^b \left\{ \frac{d\left[r M_r(\bar{\alpha}_\theta) \right]}{dr} - M_\theta(\bar{\alpha}_\theta) + \int_0^r r p(r) dr \right\} \alpha_{\theta i} r dr = 0; \; i = 1, \ldots, n \tag{4.13}$$

Equations (4.13) are n equations which determine the parameters C_i. Here, as in the Rayleigh-Ritz Method, only a restricted set of admissible curvatures approximate α_θ; therefore, the true strain condition is not obtained.

Klotter (12) proves the equivalence of the above two methods. Fox (13) (chap. 7) suggests what accuracy is obtainable by a suitable choice of assumed functions. A general explanation of these and other approximate methods is given by Sokolnikoff (14) (chap. 5). The next section demonstrates the use of Galerkin's Method.

4.4 The Numerical Example.

The numerical approximation using equation (4.13) with Galerkin's Method is demonstrated by solving the example used in sections 2.2 and 3.2. This example is a simply supported plate with a uniform load p and a parabolic curvature-moment relation. The determination of the bracketed term of formula (4.13) as a function of the assumed $\bar{\alpha}_\theta$ is performed in section 3.1. Therefore, we have

$$\int_0^b \left\{ D \left[r^2 \frac{d^2\bar{\alpha}_\theta}{dr^2} + 3r \frac{d\bar{\alpha}_\theta}{dr} - B\left(r^2 \frac{d^2\bar{\alpha}_\theta}{dr^2} + 3r \frac{d\bar{\alpha}_\theta}{dr}\right) - r\frac{dB}{dr}\left(r \frac{d\bar{\alpha}_\theta}{dr} + \frac{3}{2}\bar{\alpha}_\theta\right)\right] + \int_0^r rp\,dr \right\} \alpha_{\theta i}\, r\,dr = 0, \quad i = 1, \ldots, n. \tag{4.14}$$

where

$$B = 0.6\lambda h^2 \left[3\,\bar{\alpha}_\theta^{\,2} + 3\bar{\alpha}_\theta \left(r \frac{d\bar{\alpha}_\theta}{r}\right) + \left(r \frac{d\bar{\alpha}_\theta}{dr}\right)^2 \right]. \tag{4.15}$$

The boundary conditions are

a). At the center of the plate, $r = 0$,

$M_r = M_\theta$, and, $\alpha_{\theta i}$ is finite (4.16)

b). At the outer boundary of the plate, $r = b$,

$M_r = 0, \; r \dfrac{d\alpha_{\theta i}}{dr} + \dfrac{3}{2}\alpha_{\theta i} = 0.$

Polynomials in r^2 are selected as the approximating functions $\alpha_{\theta i}$. The polynomial for a one term approximation is parabolic. From the boundary conditions (4.16), $\alpha_{\theta 1}$ is determined as

$$\alpha_{\theta 1} = 7 - 3\,(r/b)^2 \tag{4.17}$$

The form of the elastic solution suggests that $\bar{\alpha}_\theta$ be written as

$$\bar{\alpha}_\theta = C\,\frac{pb^2}{D}\left[7 - 3\,(r/b)^2\right], \tag{4.18}$$

where C is the adjustable parameter.

An algebraic equation for C is found by evaluating equations (4.14) and (4.15) with expression (4.18). This equation is

$$Y^4C^3 - 0.01925\,C + 0.000401 = 0, \tag{4.19}$$

where $Y^2 = \lambda^{1/2}\,hpb^2/D$. The parameter Y first appeared in section 2.2 as an indication of the degree of plastic straining.

Equation (4.19) can be solved exactly, but the value of C for a given Y is found with less effort by approximate methods. The following method of Newton gives accurate results with one approximation. Formula (4.19) is expressed as a function of C.

$$f(C) = Y^4C^3 - 0.01925C + 0.000401 \tag{4.20}$$

The (n + 1)th estimate of C is denoted in terms of the n^{th} approximation plus a correction factor Δ_n.

$$C_{n+1} = C_n + \Delta_n \tag{4.21}$$

This correction factor is found with the value of the function and its derivative at C_n.

$$A_n = -f(C_n) \bigg/ \frac{df(C_n)}{dC} \qquad (4.22)$$

The combination of (4.20), (4.21), and (4.22) yields

$$C_{n+1} = \frac{Y^4 C_n^3 - 0.0002005}{1.5\, Y^4 C_n^2 - 0.00962} \qquad (4.23)$$

The values in Table V were determined with formula (4.23).

Value of the Multiplying Factor for

a One Term Strain-Energy Approximation

TABLE V

Y	C
0	.02083
0.775	0.02102
1.056	0.02148
1.368	0.02305

For the two term approximation, the second polynomial $\alpha_{\theta 2}$, which satisfies the boundary condtions (4.16), is

$$\alpha_{\theta 2} = 11(r/b)^2 - 7(r/b)^4. \qquad (4.24)$$

The two term approximation for $\bar{\alpha}_\theta$ is

$$\bar{\alpha}_\theta = \frac{pb^2}{D} \left\{ C_1 \left[7 - 3(r/b)^2\right] + C_2 \left[11(r/b)^2 - 7(r/b)^4\right] \right\} \qquad (4.25)$$

The second term, $\alpha_{\theta 2}$, alters the value of the parameter C_1 found in the one term approximation. For the two term estimate of α_θ, the numerical work increases greatly. The form of the algebraic equations for C_1 and C_2 found by using expressions (4.14) and (4.25) is

$$A_{11}C_1^{\,3} + A_{12}C_2C_1^{\,2} + A_{13}C_2^{\,2}C_1 + A_{14}C_2^{\,3} + A_{15}C_1 + A_{16}C_2 + A_{17} = 0$$

$$A_{21}C_1^{\,3} + A_{22}C_2C_1^{\,2} + A_{23}C_2^{\,2}C_1 + A_{24}C_2^{\,3} + A_{25}C_1 + A_{26}C_2 + A_{27} = 0 \tag{4.26}$$

The a_{ij} are constants which contain the parameter Y. For the determination of C_1 and C_2 with a given value of Y, two simultaneous third degree equations must be solved.

4.5 Evaluation of the Method.

A one term approximation by the minimum potential energy method for a plastically bent plate adjusts the numerical coefficient of the elastic solution. For the elastic plate the value of C in expression (4.18) is $C = 1/48 = 0.02083$. Due to the plastic behavior of the material, the circumferential curvature increases a maximum of 10.6 percent. The one term approximation gives a rapid method for directly obtaining a magnitude correction factor for the quantities connected with the curvature, but it does not show how the form of the plate is altered due to the plastic behavior of the material.

The two term approximation makes a finer adjustment of the magnitude factor and also gives some information regarding variations of the curvature with the radius. Besides the increased numerical work necessary to obtain an expression of the form (4.26), two simultaneous third degree algebraic equations must be solved for each particular example. The lack of a numerical checking procedure also presents a strong objection to higher term approximations by potential energy methods. Further comparison of this method with the methods of the previous chapters is given in Chapter VI.

CHAPTER V

THE APPLICATION OF COMPLEMENTARY POTENTIAL ENERGY

5.1 Formulation of the Minimum Complementary Potential Energy Method for the Bending of Circular Plates.

The material in Chapter IV introduces the variational procedure. The variational principle for this chapter is the principle of minimum complementary potential energy. This method is applicable when the moment-curvature relations state the curvatures in terms of the moments. Thus, an inversion of equations IV of Table I is necessary.

For this inversion a useful quantity M, similar in form to the stress invariant S, is defined as

$$M^2 = \frac{1}{3}\left(M_r^2 - M_r M_\theta + M_\theta^2\right). \tag{5.1}$$

The substitution of equations IV of Table I into (5.1) yields the relation

$$M = \frac{D(1 - B)}{2} A \tag{5.2}$$

between A and M, the quantity A can be expressed in terms of M as

$$A = \frac{2M}{D}(1 + B'), \tag{5.3}$$

where

$$B' = \sum_{n=1}^{\infty} \beta_n \left(\frac{2h}{D}\right)^{2n} M^{2n}. \tag{5.4}$$

The β_n's are constants determined by substituting equations (5.3) and (5.4) into IV and V of Table I and equating the coefficients of like powers of M. The first two coefficients are

$$\beta_1 = \frac{3}{5}\lambda_1, \ \beta_2 = 3(\frac{3}{5}\lambda_1)^2 + \frac{3}{7}\lambda_2.$$

Equations IV of Table I, (5.1), (5.2), and (5.3) are solved for the curvatures in terms of the moments:

$$\alpha_r = \frac{2(1 + B')}{D} \frac{\partial M^2}{\partial M_r},$$

$$\alpha_\theta = \frac{2(1 + B')}{D} \frac{\partial M^2}{\partial M_\theta}. \tag{5.5}$$

These are the inverted moment-curvature relations.

For the moment-curvature relations (5.5), the complementary potential energy per unit area of the plate is expressed in terms of M as

$$W_c = \int_0^{M^2} \frac{2(1 - B')}{D} dM^2. \tag{5.6}$$

The complementary potential energy for the plate is

$$\Phi_c = 2\pi \int_0^b (W_c - wp)\, rdr. \tag{5.7}$$

For the variational procedure of this section the following definitions are necessary:

1). Admissible moments are moments which satisfy equation III of Table I and the boundary conditions for the moments.

2). Admissible curvatures are curvatures derived from admissible moments and equations (5.5).

3). Admissible displacements are displacements calculated from admissible curvatures and equations I of Table I. Thus, the complementary potential energy Φ_c can be expressed in terms of only admissible moments and the given loads.

The principle of minimum complementary potential energy states that the admissible moments which render Φ_c a minimum, yield admissible curvatures that satisfy equation II of Table I. Thus, the above procedure gives the solution for the bending of a circular plate, or moments and curvatures which satisfy equations I, II, III, IV of Table I. The proof of the principle of minimum complementary potential energy follows:

A necessary condition for Φ_c to be a minimum is that the first variation of Φ_c is zero for all admissible moments. Because the loads are prescribed on the surface of the plate, the first variation of Φ_c is

$$\delta\Phi_c = 2\pi \int_0^b \left[\frac{2(1 + B')}{D} \frac{\partial M^2}{\partial M_r} \delta M_r + \frac{2(1 + B')}{D} \frac{\partial M^2}{\partial M_\theta} \delta M_\theta \right] r dr = 0. \tag{5.8}$$

The substitution of equations (5.5) into (5.8) yields

$$\int_0^b (\alpha_r \delta M_r + \alpha_\theta \delta M_\theta) \, r dr = 0. \tag{5.9}$$

The variations of the admissible moments must satisfy equation III of Table I:

$$\frac{d(r\delta M_r)}{dr} = \delta M_\theta . \tag{5.10}$$

With the substitution of the following identity

$$\begin{aligned} \int_0^b r\alpha_\theta \frac{d(r\delta M_r)}{dr} dr &= r^2 \alpha_\theta \delta M_r \Big|_0^b - \int_0^b \frac{d(r\alpha_\theta)}{dr} \delta M_r \, r dr \\ &= 0 - \int_0^b \frac{d(r\alpha_\theta)}{dr} \delta M_r \, r dr \end{aligned} \tag{5.11}$$

and equation (5.10) into (5.9) we find

$$\int_0^b \left[\alpha_r - \frac{d(r\alpha_\theta)}{dr} \right] \delta M_r \, r dr = 0. \tag{5.12}$$

When equation (5.12) is satisfied for all possible admissible variations of M_r, the bracketed term must be identically zero. Thus, equation II of Table I is satisfied and the principle of minimum complementary energy is proved.

5.2 Approximate Methods.

The methods of section 4.3 can be used to evaluate approximately the integrals (5.7) and (5.12). The integral (5.7) is handled by the Rayleigh-Ritz Method with the simplification

$$\delta \int_0^b W_c \, r dr = 0, \tag{5.13}$$

because the pressure p(r) is prescribed on the surface of the plate and does not affect the variation of Φ_c. The assumed radial moment M_r is

$$\overline{M}_r = \sum_{i=1}^{n} C_i M_{ri}(r) \tag{5.14}$$

where the $M_{ri}(r)$ satisfy the boundary conditions and the C_i are the adjustable parameters. The circumferential moment M_θ is

$$\overline{M}_\theta = \frac{d(r\overline{M}_r)}{dr} + \int_0^r rp dr, \tag{5.15}$$

which assures the satisfaction of the equilibrium conditions for admissible moments. The vanishing of the partial derivatives with respect to the n parameters C_i of expression (5.13) evaluated in terms of (5.14) and (5.15) determines the following n equations for the n arbitrary parameters C_i:

$$\frac{\partial}{\partial C_c} \int_0^b W_c \, r dr = 0. \quad i = 1, \ldots, n \tag{5.16}$$

Equations (5.16) ensure an extremium of the complementary potential energy for admissible moment variations of the restricted set (5.14); therefore, only an approximate solution is obtained.

Equation (5.12) is approximated by Galerkin's Method. Galerkin's Method gives n equations to determine the C_i of (5.14) as

$$\int_0^b \left\{ \alpha_r (\bar{M}_r) - \frac{d\left[r\alpha_\theta(\bar{M}_r) \right]}{dr} \right\} M_{ri} \, r dr = 0 \tag{5.17}$$

$$i = 1, \ldots, n$$

The curvatures in equations (5.17) are calculated in terms of $\bar{M}_r$ with the moment-curvature relations (5.5) and expressions (5.14) and (5.15). The result of this calculation is

$$\int_0^b \left[(1 + B')\left(r^2 \frac{d^2\bar{M}_r}{dr^2} + 3r \frac{d\bar{M}_r}{dr} + r^2 p + \frac{3}{2} \int_0^r rp dr\right) + r \frac{dB'}{dr} \left(r \frac{d\bar{M}_r}{dr} + \frac{\bar{M}_r}{2} + \int_0^r rp dr\right) \right] M_{ri} \, r dr = 0 \tag{5.18}$$

$$i = 1, \ldots, n.$$

These n equations are enough to determine the n parameters C_i.

5.3 The Numerical Example.

The solution of the example problem of sections 2.2, 3.2 and 4.4 is now approximated by the use of the Minimum Complementary Potential Energy Method. Formula (5.18) is used for Galerkin's approximation method. This numerical example has a parabolic moment-curvature curve when the moment invariant, M, is expressed in terms of the curvature invariant, A. The factor B' of the inverse relationship, becomes the infinite series

$$B' = 2.4 \left(\frac{\lambda^{1/2} hM}{D}\right)^2 + 17.28 \left(\frac{\lambda^{1/2} hM}{D}\right)^4 + \ldots \qquad (5.19)$$

Expression (5.19) is a specialization of (5.4) when only λ_1 is considered. An approximation to expression (5.19) of the form

$$B' = 2.4 \left(\frac{\lambda^{1/2} hM}{D}\right)^2 \qquad (5.20)$$

is made to simplify the calculations.

One justification of this estimate is that the actual moment-curvature relationship of M to A is a parabolic approximation; therefore, an approximation could be made by a parabola of the form (5.20). The factor λ can be adjusted to give as much accuracy as possible for the range of M to be considered. The adequacy of expression (5.20) as a representation of (5.19) is discussed later.

The one term polynomial approximation is suggested by the form of the elastic solution of the simply supported circular plate with constant load p.

$$M_r = Cpb^2 \left[1 - (r/b)^2\right] . \qquad (5.21)$$

Here, C is the adjustable parameter and

$$M_{r1} = 1 - (r/b)^2. \qquad (5.22)$$

The evaluation of expression (5.18) with (5.1), (5.15), (5.20), (5.21) and (5.22) yields the cubic equation for C

$$Y^4\,(C^3 - 0.7366\,C^2 + 0.1920\,C - 0.01674)$$

$$+ 0.5952\,C - 0.1302 = 0\,, \tag{5.23}$$

where $Y^2 = \lambda^{1/2}\,hpb^2/D$. The parameter Y first appeared in section 2.2 as an indication of the degree of plastic straining.

By the approximation method of Newton, which is explained in section 4.4, the successive approximations for C are obtained from

$$C_{n+1} = \frac{Y^4(C_n^{\ 3} - 0.3683\,C^2 + 0.00837) + 0.0651}{Y^4(1.5\,C_n^{\ 2} - 0.7366\,C + 0.0960) + 0.2976}\,. \tag{5.24}$$

Formula (5.24) is used to determine the values of C in Table VI.

Expression (5.20) is now compared with (5.19). The quantity $(\lambda^{1/2}hM/D)^2$ is evalutated with (5.1) and (5.21) as

$$\left(\frac{\lambda^{1/2}hM}{D}\right)^2 = \frac{Y^4}{3}\left\{C^2\left[1 - 4(r/b)^2 + 7(r/b)^4\right]\right.$$

$$\left. + \frac{C}{2}\left[(r/b)^2 - 5(r/b)^4\right] + \frac{1}{4}\,(r/b)^4\right\}. \tag{5.25}$$

Equation (5.25) has its maximum value at r = 0.

$$\left(\frac{\lambda^{1/2}hM}{D}\right)^2_{max} = Y^4C^2/3 \tag{5.26}$$

This parameter indicates the difference between (5.19) and (5.20). When Y = 1.368, the first three terms of expression (5.19) give a value of B' over 70 percent greater than (5.20). Thus for large values of Y these two expressions are not equivalent.

For a two term approximation for M_r, a similar procedure to that of section 4.4 can be applied. The same numerical difficulties arise and the extra effort involved in the calculations makes the two term approximation impractical.

Value of the Multiplying Factor for a One Term Complementary Strain-Energy Approximation

TABLE VI

Y	C
0	0.21875
0.775	0.2185
1.056	0.2178
1.368	0.2163
2.00	0.2094

5.4 Evaluation of the Method.

For the calculations of section 5.3, the moment-curvature relation is given by formulae (5.2) and (5.20). For large values of the parameter Y, the moment-curvature relation of section (5.3) is not equivalent to the relation used for the numerical calculations of the previous chapters. Thus, the numerical results of this chapter cannot be compared with those of the previous chapters. Also, the parameter Y cannot be interpreted in terms of the calculations of Chapter II. With the Y and C determined by equation (5.23) and the M given by (5.25), M is a monotonously increasing function of Y; thus, Y still indicates the stress condition in the plate. For the moment-curvature relation with a B' given by (5.20), the value of Y is not restricted, because the moment-curvature relation determined by this B' does not have a horizontal tangent.

The complementary potential energy method and the potential energy method give moments and curvatures which bracket the actual potential and complementary potential energy of the plate. A discussion of this bracketing is found in the report by Greenberg (16). In this work, the bracketing of the potential and complementary potential energy is not obtained because different moment-curvature relations are used to calculate the potential and complementary potential energy.

The elastic solution of a circular plate gives a M_r where $C = 7/32 = 0.21875$. This elastic coefficient can be compared with the values of C in Table VI which consider the plastic moment-curvature relation. The one term approximation of section 5.3 makes an adjustment for the magnitude of M_r but does not show the change in the distribution of the moments due to plastic effects. Higher term approximations adjust the distribution of the moments but lead to numerical difficulties which make them impractical.

CHAPTER VI

CONCLUSION

6.1 Comparison and Interpretation of Numerical Results.

The work of the previous chapters develops four methods for obtaining approximations of the stress and strain condition of a plastically bent circular plate. The numerical calculations of Chapter V for the minimum complementary potential energy method use a moment-curvature relation different from that of the previous chapters. This difference makes a comparison of the numerical results of Chapter V with those of the previous chapters meaningless. Therefore, only the numerical results of Chapters II, III, and IV are compared.

The circumferential curvature, α_θ, is used as the measure of comparison for the various methods. Figure 10 shows α_θ as calculated by the various methods for $Y = 1.368$ and $Y = 1.056$. These cases represent a plate which has reached a maximum strain condition at the center, $Y = 1.368$, and a plate having only half the maximum permissible strain invariant at the center, $Y = 1.056$. The results for Sokolovsky's Method are calculated from the data on Figure 9 and formula (2.32). The results for the iteration method are obtained with the use of Table IV and expression (3.21). The curve for the minimum potential energy method is obtained by using Table V and formula (4.18). The iteration method is calculated by using the two correction terms f_1 and f_2, while the potential energy method uses a one term approximation.

For $Y = 1.056$, the contribution of the f_2 term in formula (2.32) of the iteration method increases α_θ by 1 percent. Thus for small Y, the one term iteration approximation gives quick, accurate results. Actually, this approximation introduces less error for small Y than Sokolovsky's Method when the increments in the numerical integration are of the size used in section 2.3. The algebraic work for the iteration method is also less than that of the minimum potential energy method.

The plate which has reached the limit of plastic strain at the center offers the best indication of the accuracy of the different methods. The results for this plate are shown in figure 10 ($Y = 1.368$). The values for α_θ from Sokolovsky's Method and the iteration method compare favorably. The difference in these two curves indicates the magnitude of the neglected terms in formula (3.21). If additional terms are used in the approximation for $\alpha_\theta^{(n)}$ in the iteration method, this value of $\alpha_\theta^{(n)}$ would approach the value of α_θ given by Sokolovsky's Method. The values of α_θ for the minimum potential energy method lie below those of the other methods. When only the first correction term, f_1, of formula (3.21) is used, the iteration and the minimum potential energy method have close numerical agreement.

The deflection of the mean surface of the plate is calculated by using the results of the iteration method in formula (2.34). The α_θ from the iteration method is used to calculate $w(r)$ because it is expressed in the form of a polynomial which is easily integrated and has a good numerical agreement with the α_θ given by Sokolovsky's Method.

The result of the substitution of (3.21) into (2.34) is

$$w(r) = \frac{pb^2}{48D}\Big\{2.75 - 3.5(r/b)^2 + 0.75(r/b)^4 + Y^4\big[0.06873$$
$$- 0.11635(r/b)^2 + 0.07178(r/b)^4 - 0.02940(r/b)^6 + 0.00524(r/b)^8\big]$$
$$+ Y^8\big[0.01103 - 0.02275(r/b)^2 + 0.02328(r/b)^4 - 0.01883(r/b)^6$$
$$+ 0.00991(r/b)^8 - 0.00302(r/b)^{10} + 0.00038(r/b)^{12}\big]\Big\} \qquad (6.1)$$

The values for $w(r)$ in figure 11 are calculated with formula (6.1) and give the deflection of a circular plate for $Y = 1.368$, $Y = 1.056$, and for the case of linear stress-strain relation in which case $Y = 0$. The increase in the deflection at the center of the plate due to the non-linear term in the stress-strain relation for $Y = 1.368$ is about 13 percent.

The radial bending moment M_r for Sokolovsky's Method is found by substituting the values for "a" and ω of figure 9 into equation (2.33). When the a_θ given by formula (4.18) is substituted into equation IV of Table I, the M_r for the potential energy method is

$$M_r = \frac{21Pb^2C}{2}\left[1 - (r/b)^2\right]\left\{1 - \frac{5Y^4C^2}{9}\left[39(r/b)^4 - 84(r/b)^2 + 49\right]\right\}. \qquad (6.2)$$

In the example of the simply supported plate, the boundary conditions on a_θ are determined by equation IV of Table I; thus, formula (6.2) gives an M_r which satisfies the boundary condition, $M_r = 0$ at $r = b$. In general, for the minimum potential energy method, the admissible curvatures must satisfy the boundary conditions, but the derived bending moments do not have to satisfy the boundary conditions. Figure 12 gives a comparison of the radial bending moment calculated from equations (2.33) and (6.2) for the fully strained plate, where $Y = 1.368$.

6.2 Remarks on the Theory of Plastic Flow.

In section 1.4 the stress-strain relations of the secant modulus theory (the theory of plastic deformations) are formulated. The combination of (1.13) and (1.14) gives

$$s_i = 2G\left[1 - f(E)\right] e_i, \ i = 1,2,3. \tag{6.3}$$

Inverse relations relating the total strains to the stresses are found by a procedure similar to that of section 5.1. These inverse relations are of the form

$$e_i = \frac{1}{2G}\left[1 + g\,(S)\right] s_i, \ i = 1,2,3. \tag{6.4}$$

Expressions (6.3) and (6.4) are valid for the case of loading only. They relate the final stress condition to the final strain condition of the element independently of the path of loading.

Recent experimental investigations by Phillips (20) and Morrison and Shepard (21) indicate that the plastic behavior of a metal is described by the laws of plastic flow. Hill (7) (chap. 2) and Prager and Hodge (18) (chap. 1) give a formulation of the theory of plastic flow. In what follows we shall consider the case of loading only.

The theory of plastic flow relates the increments of the strain deviation to 1). the increments of the stress deviation and 2). the stress invariant by

$$de_i = \frac{1}{2G}\left[ds_i + s_i h(S)dS\right], \ i = 1,2,3. \tag{6.5}$$

Thus the total strain deviations are given by the line integrals

$$e_i = \frac{1}{2G}\left[s_i + \int_0^{S(s_i)} s_i h(S) dS \right] , \quad i = 1,2,3, \tag{6.6}$$

which depend upon the manner of stressing the element. The appearance of the line integrals in (6.6) makes the inversion of the stress-strain relations impossible. Hill (7) (chap. 2) shows how to express the increments of the stress deviations in terms of the stresses and the increments of the strain deviations as

$$ds_i = 2G\left[de_i - \left(\frac{s_i h(S) \sum_{i=1}^{3} s_i de_i}{1 + 2Sh(S)} \right) \right], \quad i = 1,2,3. \tag{6.7}$$

Equation (6.7) does not determine the increments of the stress deviations in terms of the strains only, but also assumes knowledge of the stress condition of the element.

Because the stress condition of the body must be known to determine increments in the stresses for given increments in the strains, a solution for the bending of a plate which satisfies assumption c), section 1.3, must be a step by step procedure. For this procedure, increments in the curvatures determine increments in the strain deviations which satisfy assumption c). These strain deviations together with the knowledge of the stress condition from the previous step, can be used to calculate the increments of the stresses from equation (6.7). These stress increments are used to determine increments of the bending moments which must satisfy the equilibrium equation III of Table I. The numerical work of such a procedure is enormous.

An estimate is now made of the change in the numerical results of the previous chapters if the laws of plastic flow were used instead of the laws of plastic deformations. For this estimate, figure 13 is constructed by the procedure similar to that of section 2.1 for the construction of figure 5b. Then, figure 13 is used as an aid to differentiate between the laws of plastic flow and plastic deformations.

For the construction of figure 13, the strain deviations are divided into

$$e_i = e_i^{\,e} + e_i^{\,p} \qquad i = 1,2,3, \tag{6.8}$$

where $e_i^{\,e}$ is the elastic strain deviation given by

$$e_i^{\,e} = s_i/2G \qquad i = 1,2,3, \tag{6.9}$$

for both (6.4) and (6.5). The plastic strain deviation $e_i^{\,p}$ is given by

$$e_i^{\,p} = \frac{s_i g(S)}{2G} \qquad i = 1,2,3, \tag{6.10}$$

for the theory of plastic deformations, and by

$$de_i^{\,p} = \frac{s_i h(S)\, dS}{2G} \qquad i = 1,2,3, \tag{6.11}$$

for the theory of plastic flow. Because $e_i^{\,e}$ is the same for both theories, only $e_i^{\,p}$ is compared to determine the difference between the theories of plastic deformations and plastic flow.

In figure 13a, axes $(\sqrt{3}\, e_1 p/2)$, $(\sqrt{3}\, e_2 p/2)$, and $(\sqrt{3}\, e_3 p/2)$ lie in the principal stress axes σ_1, σ_2, and σ_3. The angle ω is determined by a vector $\vec{S}$, where $\vec{S}$ has the components $s_1/\sqrt{2}$, $s_2/\sqrt{2}$, and $s_3/\sqrt{2}$. The vector $\vec{S}$ lies in the octahedral plane and is equal in magnitude to the stress invariant S.

The surface shown in figure 13b is a surface of revolution about the S axis which determines the strain hardening characteristics of the material. The S axis is perpendicular to the octahedral plane described above. For the theory of plastic deformations, the value of E^p

$$E^p = (1/2 \sum_{i=1}^{3} e_i{}^p e_i{}^p)^{1/2} \tag{6.12}$$

is the radius of the circle formed by the intersection of a plane parallel to the octahedral plane and a distance S from the octahedral plane. The substitution of equations (6.10) into (6.12) yields

$$E^p = \frac{Sg(S)}{2G} , \tag{6.13}$$

where equation (6.13) determines the curve which generates the strain hardening surface. The orthogonal projections of a vector E^p, at the angle ω in the octahedral plane, on the axes, $(\sqrt{3}\ e_1 p/2)$, $(\sqrt{3}\ e_2 p/2)$, and $(\sqrt{3}\ e_3 p/2)$ are the plastic strain deviations given by equations (6.10).

For the theory of plastic flow, expressions (6.11) are used to formulate the quantity

$$\sqrt{1/2 \sum_{i=1}^{3} de_i{}^p de_i{}^p} = \frac{Sh(S)}{2G}\, dS. \tag{6.14}$$

This quantity is the increment in the radii of the circles formed by the intersections of the planes parallel to the octahedral plane at the distances S and S + dS with the strain hardening surface. Equation (6.14) determines the shape of the curve which generates the strain hardening surface. The orthogonal projections of the incremental vector

$\sqrt{1/2 \sum_{i=1}^{3} de_i de_i}$, at the angle ω in the octahedral plane, on the axes $(\sqrt{3}\, e_1 p/2)$, $(\sqrt{3}\, e_2 p/2)$, and $(\sqrt{3}\, e_3 p/2)$ are the increments of plastic strain deviations given by equations (6.11).

As a simplification, the principal axes are assumed to remain fixed throughout loading. This assumption is true for the case of circular plates symmetrically loaded. Also, the strain hardening surface is assumed the same for the theory of plastic flow and for the theory of plastic deformations. With these assumptions, and if ω = const. during loading, (increasing S), the same total plastic strain deviations are predicted by both of the above theories. But when ω changes during loading, the value of the total plastic strain deviations given by the theory of flow will depend upon how ω changes, while the total plastic strain deviations given by the theory of plastic deformations depend upon the final value of ω. Thus, for any case of loading of an element where the angle ω changes during loading, the values of the total plastic strain deviations predicted by the theory of plastic deformations differ from those predicted by the theory of plastic flow. However, when the total change of ω is slight, this difference is small enough to be neglected.

The variable ω introduced in section 2.1 was determined by the direction of a vector $\vec{A}$ in the octahedral plane. Because the components of $\vec{A}$ are proportional to the components of $\vec{E}$, the vector $\vec{A}$ has the same direction as the vector $\vec{E}$. This vector $\vec{E}$ is parallel to the vector $\vec{S}$ for the laws of plastic deformations; thus, the variable ω used in section 2.1 is the same as the above ω. An inspection of figure 9 shows that the change in ω is only a few degrees at any given value of r/b during the loading of the plate; hence, the moments predicted by the theory of plastic deformations should be very close to those predicted by the theory of plastic flow, but not identical.

6.3 The Application of the Methods of Chapters III, IV, and V to the Bending of Rectangular Plates.

The small deflection theory of rectangular plates can be treated with the methods of Chapters III, IV, and V. A brief outline of the application of these methods is now given. Swida (19) develops the kinematic and equilibrium equations. Ilyushin (9) applies the method of minimum potential energy and the iteration method to the problem of the bending of rectangular plates.

Some of the notation of the previous chapters is now altered. The strains are defined as

$$\varepsilon_{ij} = \frac{1}{2}\left(\partial u_i \Big/ \partial x^i + \partial u_j \Big/ \partial x^i\right) , \quad i = x,y,z \tag{6.15}$$

The kinematic condition that normals to the neutral surface remain normal to the neutral surface in the deflected state still holds and allows the significant strains to be expressed as

$$\varepsilon_{xx} = - zw_{xx} \; ; \quad \varepsilon_{yy} = - zw_{yy} \; ; \quad \varepsilon_{xy} = - zw_{xy} \tag{6.16a}$$

where

$$w_{xx} = \partial^2 w/\partial x^2 \; ; \quad w_{yy} = \partial^2 w/\partial y^2 \; ; \quad w_{xy} = \partial^2 w/\partial x \partial y \tag{6.16b}$$

and w is the deflection of the neutral surface. Equations (6.16b) are the kinematic equations which express the quantities, w_{xx}, w_{yy}, and w_{xy}, in terms of the deflection w.

In the rectangular plate theory ε_{xz}, ε_{yz}, and σ_z are assumed to be zero. For the following formulation, the material is assumed incompressible, that is

$$\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} = 0. \tag{6.17}$$

From the above assumptions, the stresses are written in terms of the strains as

$$\begin{aligned} \sigma_x &= 4G\left[1 - F(E)\right](\varepsilon_{xx} + \varepsilon_{yy}/2) \\ \sigma_y &= 4G\left[1 - F(E)\right](\varepsilon_{yy} + \varepsilon_{xx}/2) \\ \tau_{xy} &= 2G\left[1 - F(E)\right]\varepsilon_{xy}. \end{aligned} \tag{6.18}$$

The second strain invariant E is

$$E^2 = \frac{1}{2}\sum_{i,j} \varepsilon_{ij}\varepsilon_{ij} = \varepsilon_{xx}^2 + \varepsilon_{yy}^2 + \varepsilon_{xx}\varepsilon_{yy} + \varepsilon_{xy}^2, \tag{6.19}$$

and as in Chapter II, E can be written in terms of a quantity A as

$$E^2 = z^2A^2 \tag{6.20}$$

where

$$A^2 = w_{xx}^2 + w_{yy}^2 + w_{xx}w_{yy} + w_{xy}^2. \tag{6.21}$$

The function $F(E)$, which expresses the plastic properties of the material, is given by the polynomial

$$F(E) = \sum_{n=1}^{\infty} \lambda_n E^{2n} = \sum_{n=1}^{\infty} \lambda_n z^{2n} A^{2n}. \tag{6.22}$$

The definitions of the bending moments

$$M_x = \int_{-h}^{h} \sigma_z zdz \;;\; M_y = \int_{-h}^{h} \sigma_y zdz \;,\; M_{xy} = \int_{-h}^{h} \tau_{xy} zdz \tag{6.23}$$

are combined with equations (6.16), (6.18), and (6.22) to yield

$$M_x = -D(1 - B)\,(w_{xx} + w_{yy}/2),$$

$$M_y = -D(1 - B)\,(w_{yy} + w_{xx}/2), \qquad (6.24)$$

$$M_{xy} = -D(1 - B)\,w_{xy}/2,$$

where

$$D = 8G\,h^3/3 \qquad (6.25)$$

$$B = \sum_{n=1}^{\infty} \frac{3\lambda_n}{2n+3}\,(hA)^{2n}.$$

The equilibrium equation for a rectangular plate is

$$\frac{\partial^2 M_x}{\partial x^2} + 2\,\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p(x,y) = 0, \qquad (6.26)$$

where p(x,y) is the distributed load per unit area which acts normal to the surface of the plate. Equations (6.16b), (6.24), and (6.26) are the equations which are used to find the solution for the bending of a rectangular plate.

A formula which can be used for the iteration method is obtained by the substitution of equations (6.16) and (6.24) into (6.26). These substitutions yield

$$\nabla^4 w = \frac{1}{D}\,(p(x,y) + \bar{p}_p), \qquad (6.27)$$

where

$$\nabla^4 w = \frac{\partial^4 w}{\partial x^4} + 2\,\frac{\partial^4 w}{\partial x^2 dy^2} + \frac{\partial^4 w}{\partial y^4}$$

and

$$p_p = D \; B\left[\nabla^4 w + (w_{xx} + w_{yy}/2)\frac{\partial^2 B}{\partial x^2} + w_{xy}\frac{\partial^2 B}{\partial x \partial y} + (w_{yy} + w_{xx}/2)\frac{\partial^2 B}{\partial y^2}\right]. \tag{6.28}$$

p_p is a pseudo-loading due to the non-linearity of the stress-strain relations. If the iteration method of Chapter III were followed, a Green's Function for the elastic plate would be developed so the effect of any arbitrary loading on the deflection w could be found by evaluating an integral. In order to find this Green's Function, the elastic plate problem of a concentrated load acting at any arbitrary point on the plate with the given boundary conditions must be solved. Such solutions are given in terms of infinite series which make the evaluation of p_p very lengthy. A direct approach, which involves less numerical effort, is to first assume $B = 0$ and directly solve the elastic problem

$$\nabla^4 w_e = \frac{p(x,y)}{D} \tag{6.29}$$

for the given boundary conditions. The w_e, found above, is then used to evaluate p_p with (6.28). This p_p is used to obtain a first approximation for w with

$$\nabla^4 w_1 = \frac{1}{D}\,(p(x,y) + p_p). \tag{6.30}$$

In general, analytic solutions of (6.29) and (6.30) are difficult to find, but relaxation or methods of finite differences can be used to get numerical results. The w found above can be used to evaluate a new p_p for a second approximation of w. Judging from the numerical results found in Chapter III, only two iterations should give sufficient accuracy.

The minimum potential energy method of Chapter IV can be used to obtain an approximate solution for the bending of a rectangular plate. From the combination of (6.21) and (6.24), we find the moment-curvature relations in terms of A as

$$M_x = \frac{-D(1-B)}{2}\frac{\partial^2 A^2}{\partial w_{xx}}, \quad M_y = \frac{-D(1-B)}{2}\frac{\partial^2 A^2}{\partial w_{yy}},$$

$$2M_{xy} = \frac{-D(1-B)}{2}\frac{\partial^2 A^2}{\partial w_{xy}}. \tag{6.31}$$

For the moment-curvature relations (6.31), the strain energy per unit area of the plate is

$$W = \int_0^{A^2} \frac{D(1-B)}{2}\, dA^2. \tag{6.32}$$

The potential energy for the plate is

$$\Phi = \iint_S (W - pw)\,dxdy. \tag{6.33}$$

The first variation of Φ must be zero for minimum potential energy. When equations (6.31) are substituted into $\delta\Phi = 0$, the result is

$$\delta\Phi = \iint_S M_x(-\delta w_{xx}) + 2M_{xy}(-\delta w_{xy}) + M_y(-\delta w_{yy}) - p\delta w\, dxdy = 0 \tag{6.34}$$

Expression (6.34) can be put in an alternate form which shows that the minimization of the potential energy for admissible curvatures satisfying (6.16b) and deflections satisfying the given boundary conditions yield moments which fulfill the equilibrium conditions.

To do this, integrations by parts of the type

$$\iint_S M_x(-\delta w_{xx}) = \int_{\text{Boundary}} M_x \left(-\delta \frac{dw}{dx}\right) ds + \iint_S \frac{\partial M_x}{\partial x} \delta \frac{\partial w}{\partial x} dxdy$$

$$= 0 + \int_{\text{Boundary}} \frac{\partial M_x}{\partial x} \delta w ds - \iint_S \frac{\partial^2 M_x}{\partial x^2} \delta w dxdy = 0 - \iint_S \frac{\partial^2 M_x}{\partial x^2} \delta w dxdy$$

(6.35)

are performed. Then equation (6.34) becomes

$$\iint_S \left(\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + p \right) \delta w dxdy = 0 \tag{6.36}$$

The bracketed term of (6.36) is the equilibrium condition (6.26).

An approximate solution is obtained by assuming a form of the deflection

$$\bar{w} = \sum_{i=1}^{n} C_i w_i(x,y) \tag{6.37}$$

where the C_i are adjustable parameters and the w_i satisfy the given boundary conditions of w. The C_i can be evaluated by the Rayleigh-Ritz method where Φ is evaluated in terms of $\bar{w}$ and the n equations

$$\frac{\partial \Phi}{\partial C_i} = 0 \qquad i = 1, \ldots, n \tag{6.38}$$

determine the C_i. The equivalent method of Galerkin may be used with equation (6.36).

The minimum complementary potential energy method of Chapter V can be applied to a rectangular plate for approximate solutions. Expressions (6.24) determine a quantity M as

$$M^2 = \frac{1}{3}(M_x^2 + M_y^2 - M_xM_y + 3M_{xy}^2) = \left[\frac{D(1-B)}{2}\right]^2 A^2 \tag{6.39}$$

As in section (5.1), A is expressed in terms of M as

$$A = \frac{2}{D}(1 + B')M \tag{6.40}$$

where

$$B' = \sum_{n=1}^{\infty} \beta_n \left(\frac{2h}{D}\right)^{2n} M^{2n} \tag{6.41}$$

The β_n have the same values as in section 5.1. Inverse moment-curvature relations are obtained from (6.24), (6.39), and (6.40) as

$$\begin{aligned} - w_{xx} &= \frac{4}{3D(1 - B)}(M_x - M_y/2) = \frac{2(1 + B')}{D}\frac{\partial M^2}{\partial M_x}, \\ - w_{yy} &= \frac{4}{3D(1 - B)}(M_y - M_x/2) = \frac{2(1 + B')}{D}\frac{\partial M^2}{\partial M_y}, \\ - w_{xy} &= \frac{2}{D(1 - B)} M_{xy} = \frac{(1 + B')}{D}\frac{\partial M^2}{\partial M_{xy}}. \end{aligned} \tag{6.42}$$

For the moment-curvature relations (6.42), the complementary strain energy per unit area of the plate is

$$W_c = \int_0^{M^2} \frac{2(1 + B')}{D}\, dM^2, \tag{6.43}$$

and the complementary potential energy of the plate is

$$\Phi_c = \iint_S (W_c - pw)dxdy. \tag{6.44}$$

The first variation of Φ_c must be zero for minimum complementary potential energy.

For the computational work, both M_x and M_y can be assumed as

$$\bar{M}_x = \sum_{i=1}^{n} C_i M_{xi}(x,y) \; ; \; \bar{M}_y = \sum_{i=1}^{n} D_i M_{yi}(x,y), \tag{6.45}$$

where C_i and D_i are parameters, and M_{xi} and M_{yi} satisfy the given boundary conditions on the moments. The twisting moment is calculated from (6.26) as

$$\bar{M}_{xy} = -\frac{1}{2}\left[\iint p(x,y)\,dxdy + \sum_{i=1}^{n} C_i \int \frac{\partial M_{xi}}{\partial x}\,dy + \sum_{i=1}^{n} D_i \int \frac{\partial M_{yi}}{\partial y}\,dx\right] + F(x) + G(y), \tag{6.46}$$

where $F(x)$ and $G(y)$ are chosen so that they give the correct boundary conditions for $\bar{M}_{xy}$. When the load is prescribed, δp is zero; thus, the term wp can be disregarded in the calculation of Φ_c because it has no effect on the first variation of Φ_c. Φ_c is calculated with equations (6.39), (6.43), (6.44), (6.45) and (6.46); then the C_i and D_i can be obtained from the 2n equations

$$\frac{\partial \Phi_i}{\partial C_i} = 0 \quad \text{and} \quad \frac{\partial \Phi_c}{\partial D_i} = 0. \tag{6.47}$$

This section has developed methods for the approximate solution of the small deflection theory of rectangular plates which parallel those for the small deflection theory of circular plates. Similar solutions for the large deflection theory of plates, where the stretching of the neutral surface is considered are not readily obtainable. Even the elastic solution of such problems gives non-linear equations which are difficult to solve, and as we have seen the elastic solution is a first step in the iteration process. In addition, due to the terms from the stretching of the neutral surface, the strain invariant E cannot be expressed as a function of the curvatures only. Therefore, the moments and curvatures cannot be the only variables in the formulization of the differential equations. Because of these mathematical difficulties, the condition of a plate is usually approximated by either the small deflection theory of bending, or a large deflection theory where only membrane stresses and strains are considered.

BIBLIOGRAPHY

1. P. M. Naghdi, "The Bending of Elastoplastic Circular Plates with Large Deflection," Journal of Applied Mechanics, Vol. 19, No. 3, pp 293-300, Sept. 1952.

2. A. Gleyzal, "Plastic Deformation of a Circular Diaphragm Under Pressure," Journal of Applied Mechanics, Vol. 15, No. 3, September 1948.

3. W. Mostow, "Plastic Deformation of Thin Plates under Hydrostatic Pressure," in Plastic Deformation, edited by H. H. Housner, Mapleton House, 1948.

4. R. Hill, "A Theory of the Plastic Bulging of a Metal Diaphragm by Laterial Pressure," The Philosophical Magazine, No. 322, November 1950.

5. S. Timoshenko, Theory of Plates and Shells, McGraw-Hill, New York, 1940.

6. V. V. Sokolovsky, Theory of Plasticity, (Russian), State Publishing House of Technical and Theoretical Literature, Moscow, 2nd Ed., 1950.

7. R. Hill, The Mathematical Theory of Plasticity, Oxford, London, 1950.

8. J. Scarborough, Numerical Mathematical Analysis, John Hopkins Press, Baltimore, 2nd Ed., 1950.

9. A. A. Ilyushin, "Some Problems in the Theory of Plastic Deformation," Prikladnaia Matematika i Mekhanika, No. 7, pp 245-272, 1943, or Plastic Deformation, edited by H. Hausner, Mapleton House, 1950.

10. Aris Phillips, "Variational Principles in the Theory of Finite Plastic Deformations," Quarterly of Applied Mathematics, Vol. VII, No. 1, pp 110-114, 1949.

11. C. Lanczos, The Variational Principles of Mechanics, University of Toronto Press, 1949.

12. K. Klotter, "Non-Linear Vibration Problems Treated by the Averaging Method of W. Ritz," Contract N6onr-251, Task Order 2 (NR-041-943), Technical Report No. 17, May 1951.

13. C. Fox, An Introduction to the Calculus of Variations, Oxford University Press, London, 1950.

14. I. Sokolnifoff, Mathematical Theory of Elasticity, McGraw-Hill, New York, 1946.

15. V. M. Panferov, "On the Convergence of a Method of Elastic Solutions for the Problem of Elastic Plastic Bending of a Plate," Prikladnaia Matematicka i Mekhanika, No. 16, pp. 195-212, 1952.

16. H. J. Greenberg, "On the Variational Principles of Plasticity," Contract N7 onr-358, Task Order 1, (NR-041-032), Graduate Division of Applied Mathematics, Brown University, Providence, R. I., March, 1949.

17. D. Trifan, "On the Plastic Bending of Circular Plates under Uniform Transverse Loads," Quarterly of Applied Mathematics, No. 7, pp. 417-427, 1948.

18. W. Prager and P. Hodge, Theory of Perfectly Plastic Solids, John Wiley and Sons, Inc., 1951.

19. W. Swida, "Die Plattengluchungen fürden Elastisch-Plastischen Zustand," Z.A.M.M., pp. 375-381, 1949-1950.

20. A. Phillips, "Combined Tension-Torsion Test for Aluminum Alloy 2S-0," Journal of Applied Mechanics, Vol. 19, No. 4, pp. 496-500, December, 1952.

21. J. L. Morrison and W. M. Shepherd, "An Experimental Investigation of Plastic Stress-Strain Relations," Inst. of Mechanical Engineers and Applied Mechanics, W.E.P. No. 55, pp 1-17, 1950.

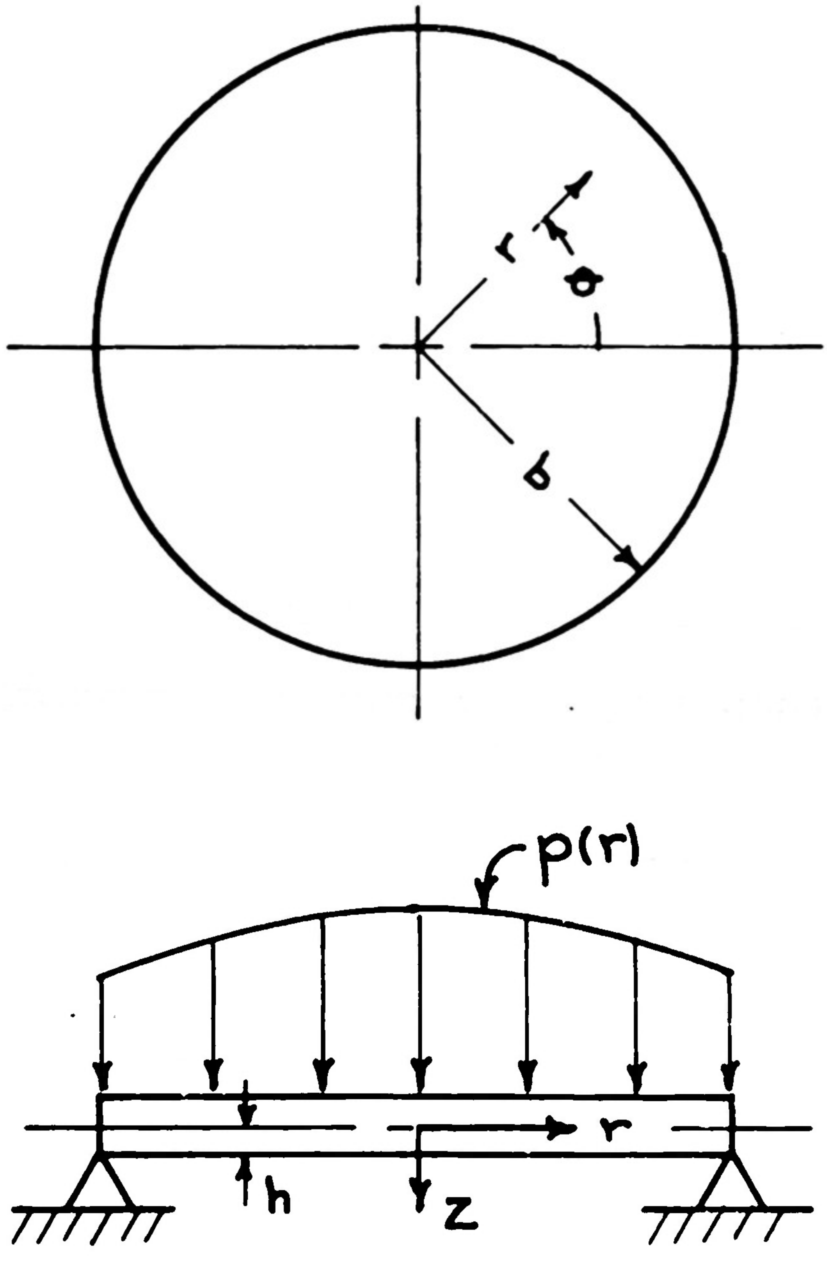


FIGURE I

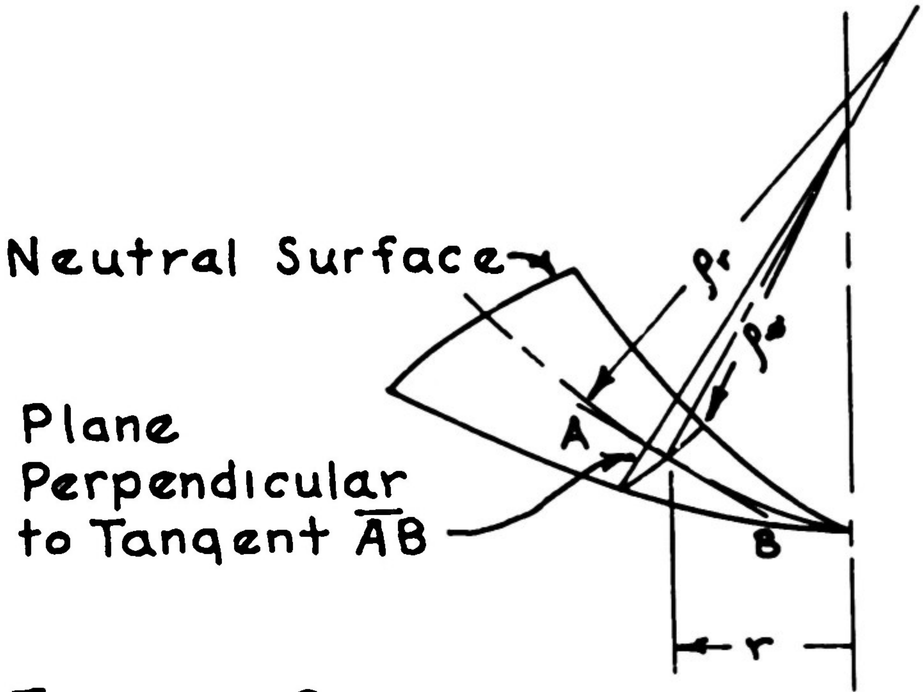


FIGURE 2

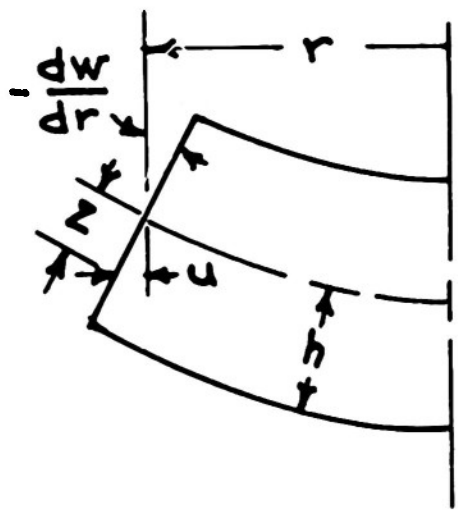


FIGURE 3

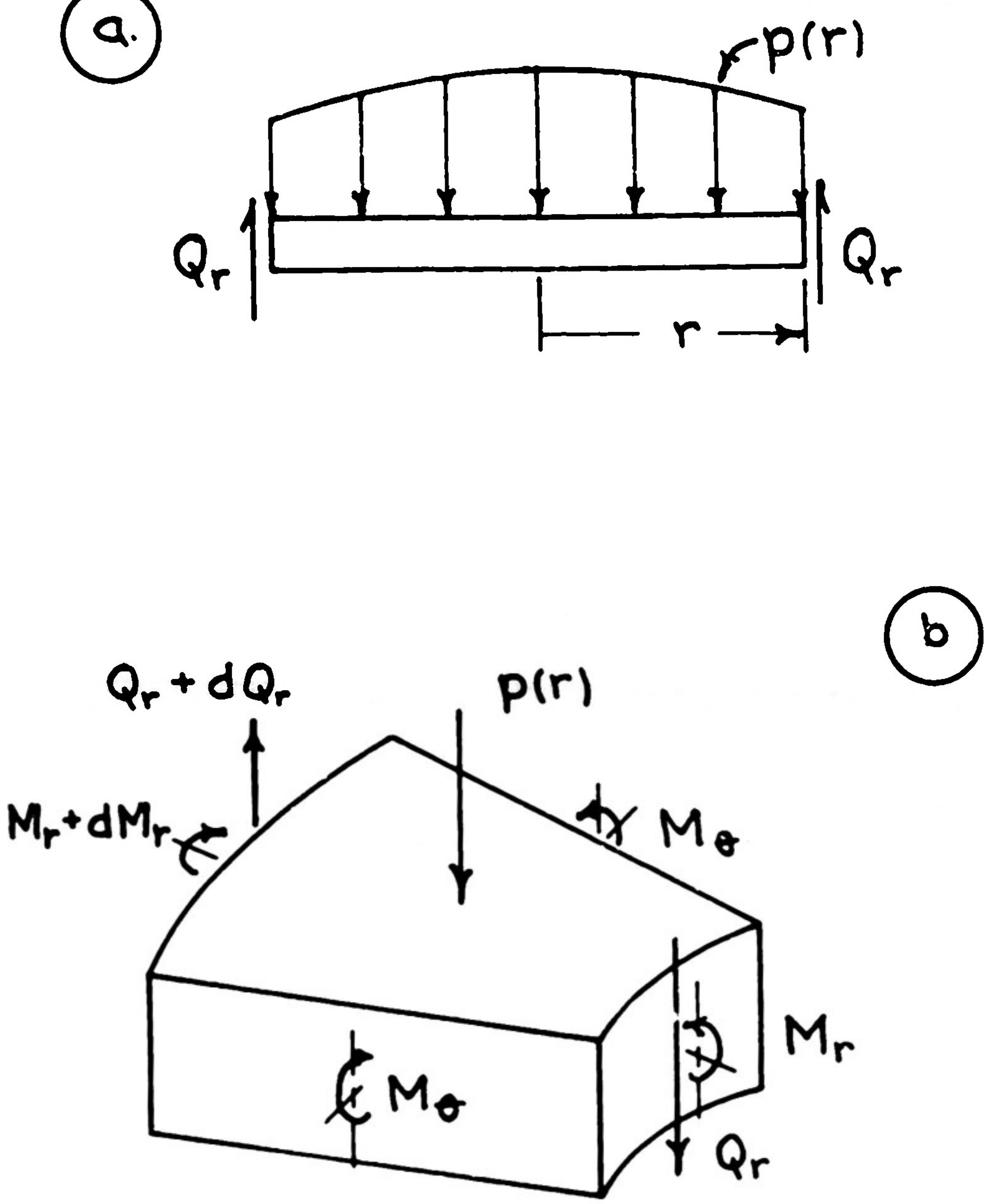
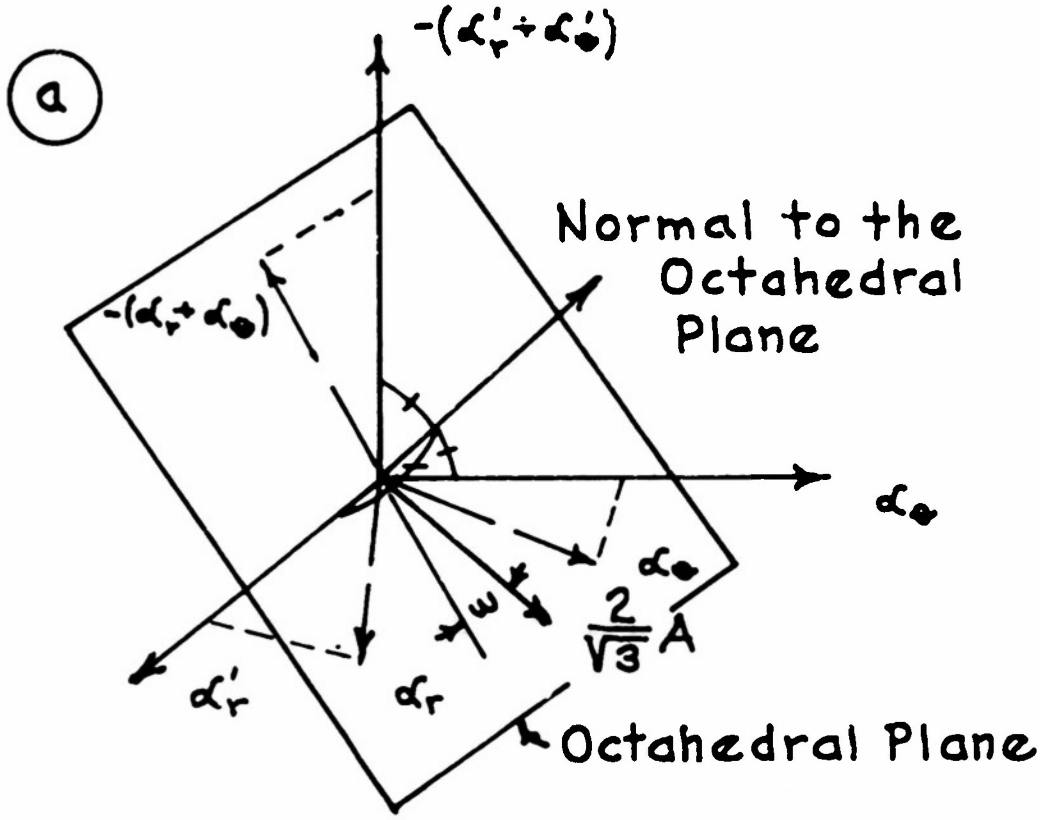


FIGURE 4

GRAPHICAL REPRESENTATION OF THE VARIABLES A AND ω



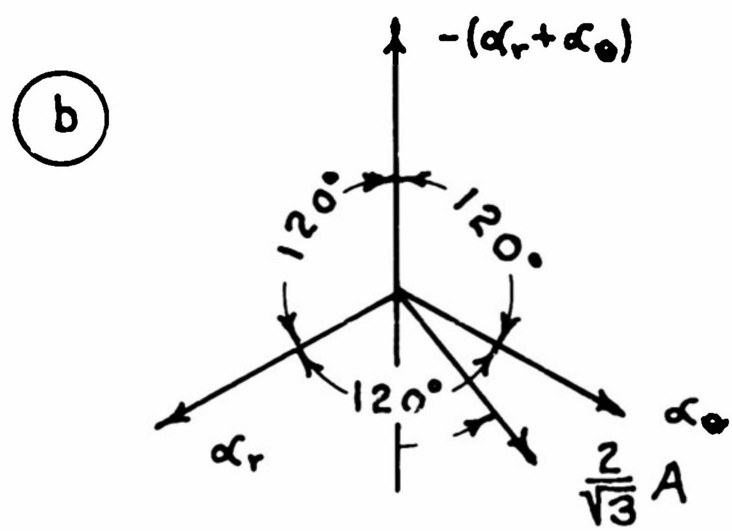


FIGURE 5

STRESS-STRAIN DIAGRAM

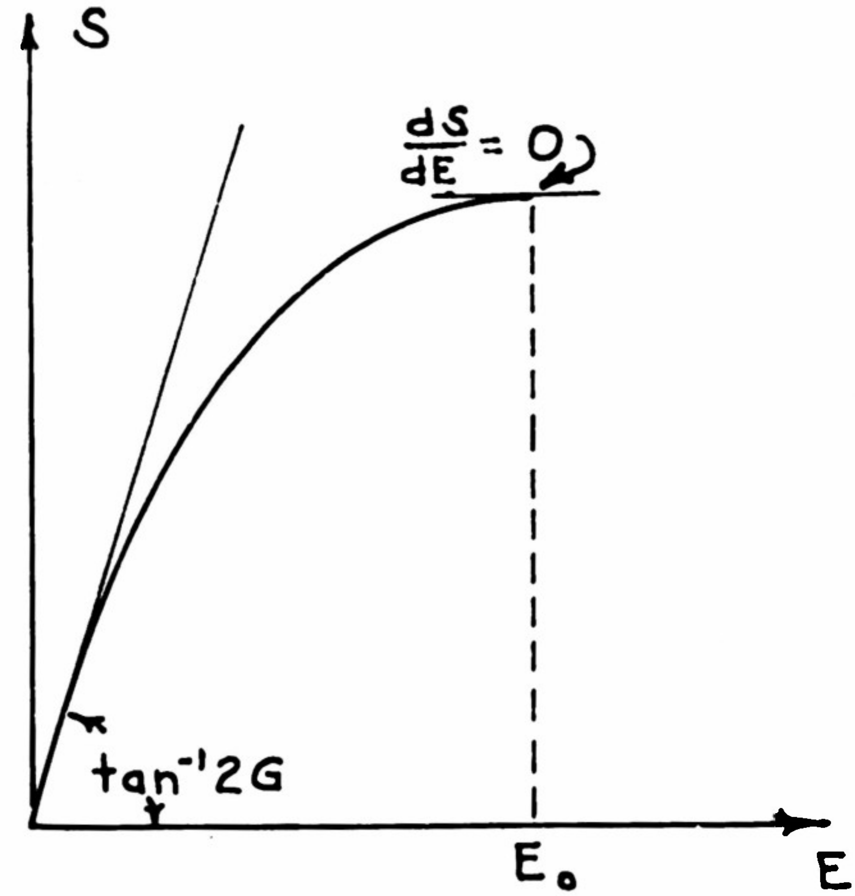
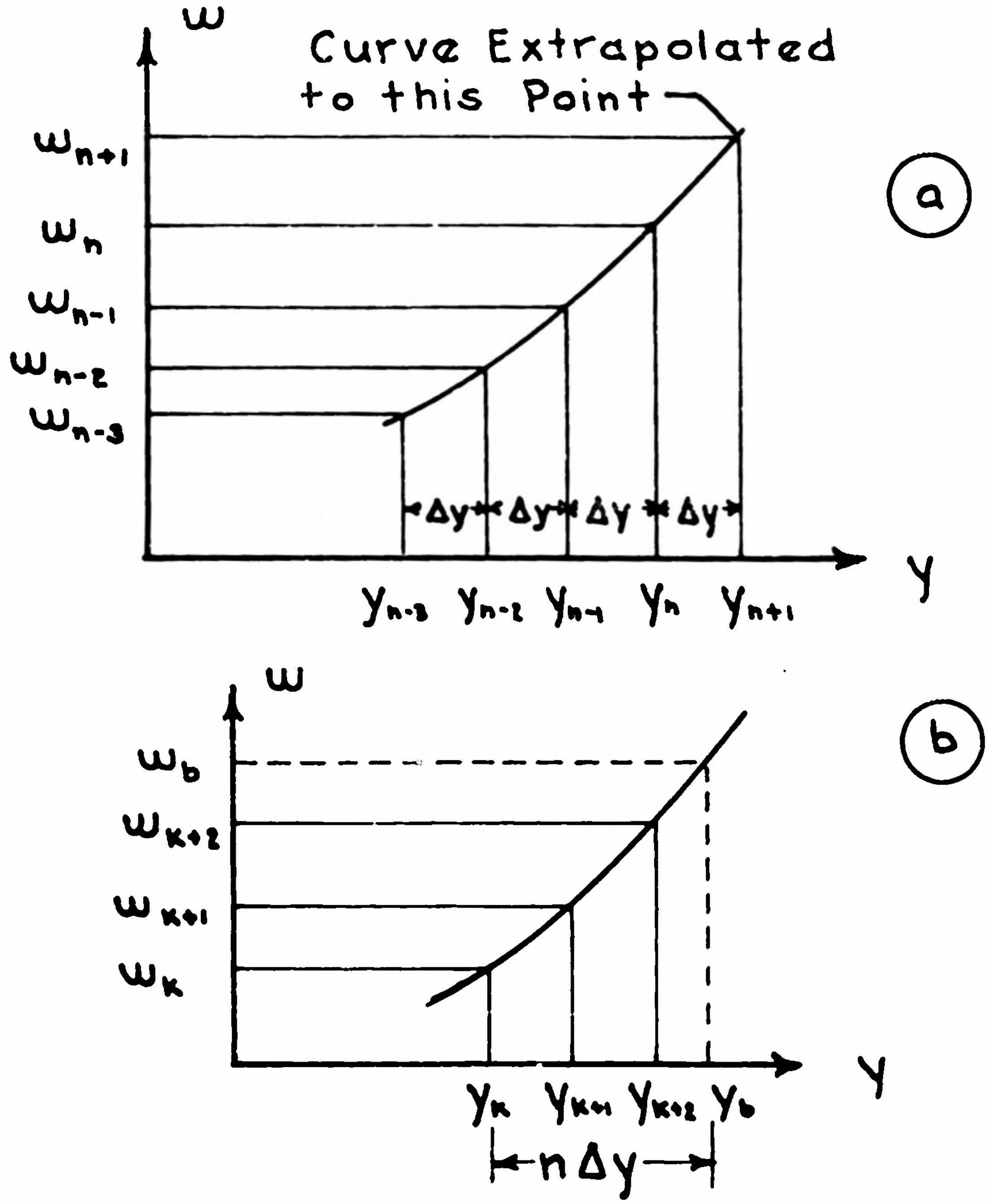


FIGURE 6



Curve Extrapolated to this Point
a
b
w
y
w_{n+1}
w_n
w_{n-1}
w_{n-2}
w_{n-3}
Δy
y_{n-3} y_{n-2} y_{n-1} y_n y_{n+1}
w_b
w_{k+2}
w_{k+1}
w_k
y_k y_{k+1} y_{k+2} y_b
$n\Delta y$

FIGURE 7

Variation of "a" and "w" with y for the Plastic Bending of a Simply Supported Circular Plate

$a = A/A_0$

$y^2 = \lambda^{1/2} h p r^2 / D$

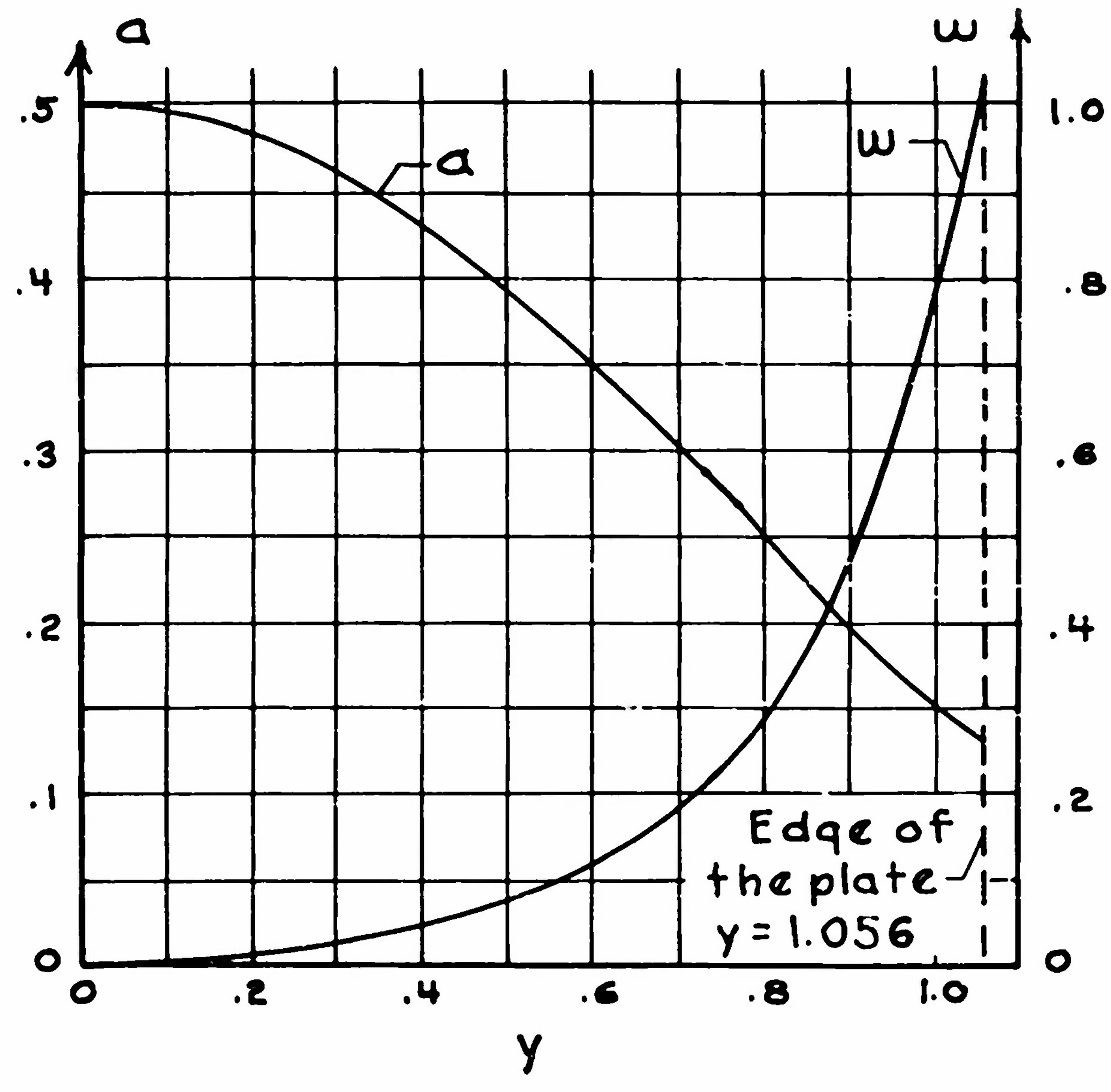
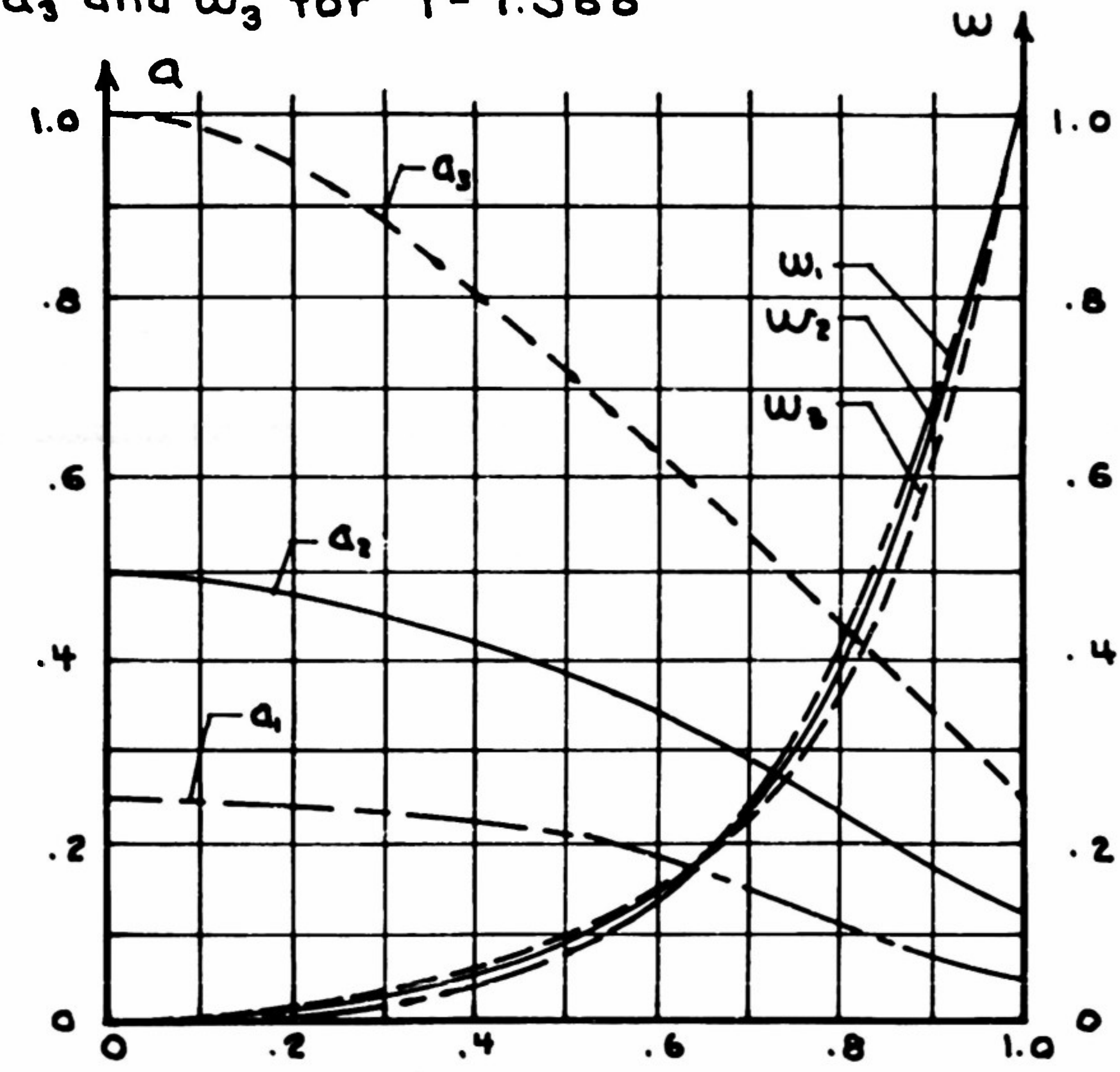


FIGURE 8

Comparison of the Variations of "a" and "w" with the Radius, r, for Various Values of the Parameter Y

a_1 and w_1 for Y = .775
a_2 and w_2 for Y = 1.056
a_3 and w_3 for Y = 1.368



r/b ratio
b = outer radius

FIGURE 9

Comparison of the Numerical Results for the Circumferential Curvature, κ_ϕ, for the Methods of Chapters II, III, and IV.

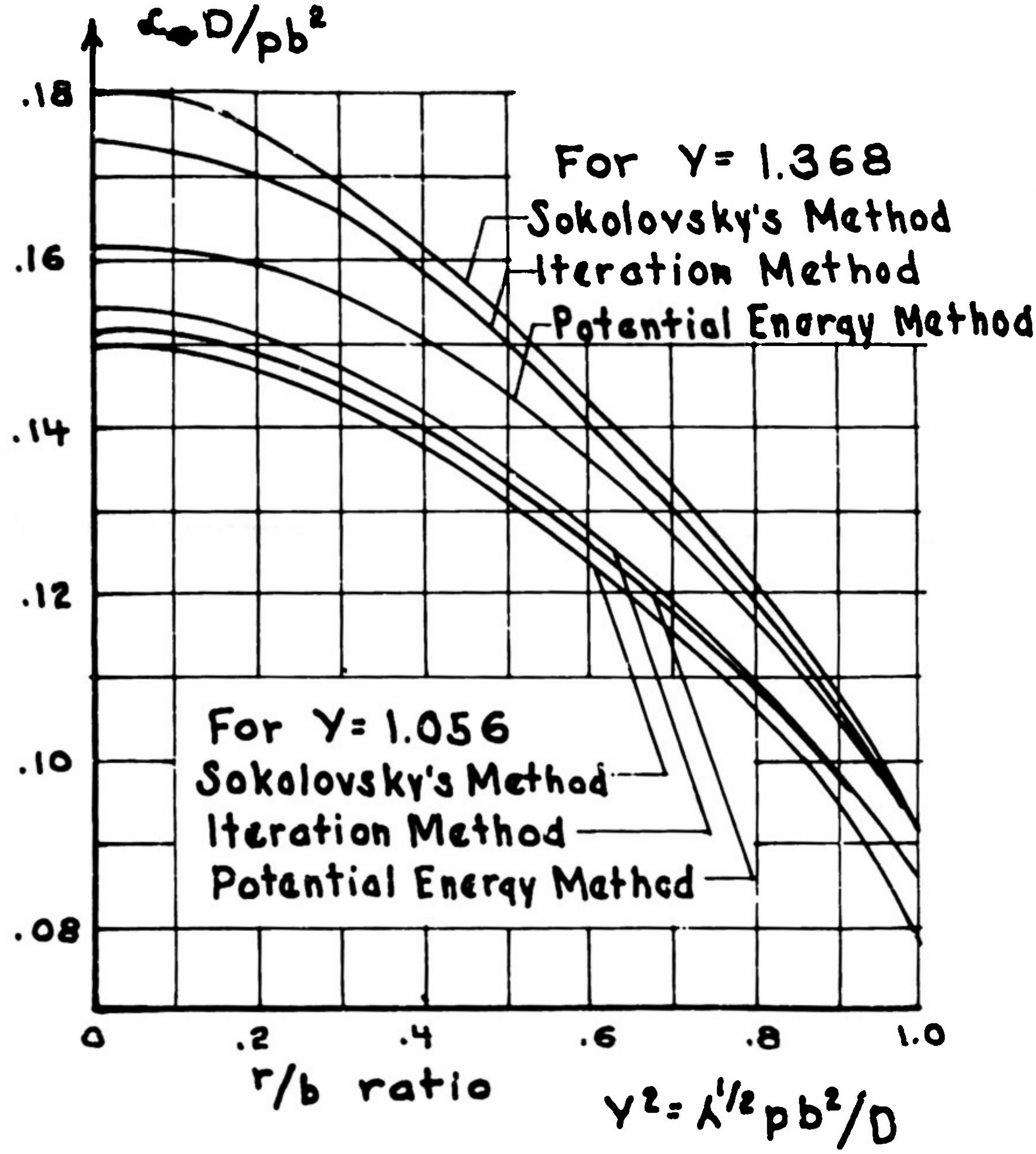


FIGURE 10

Variation of the deflection, w, with the Radial Distance

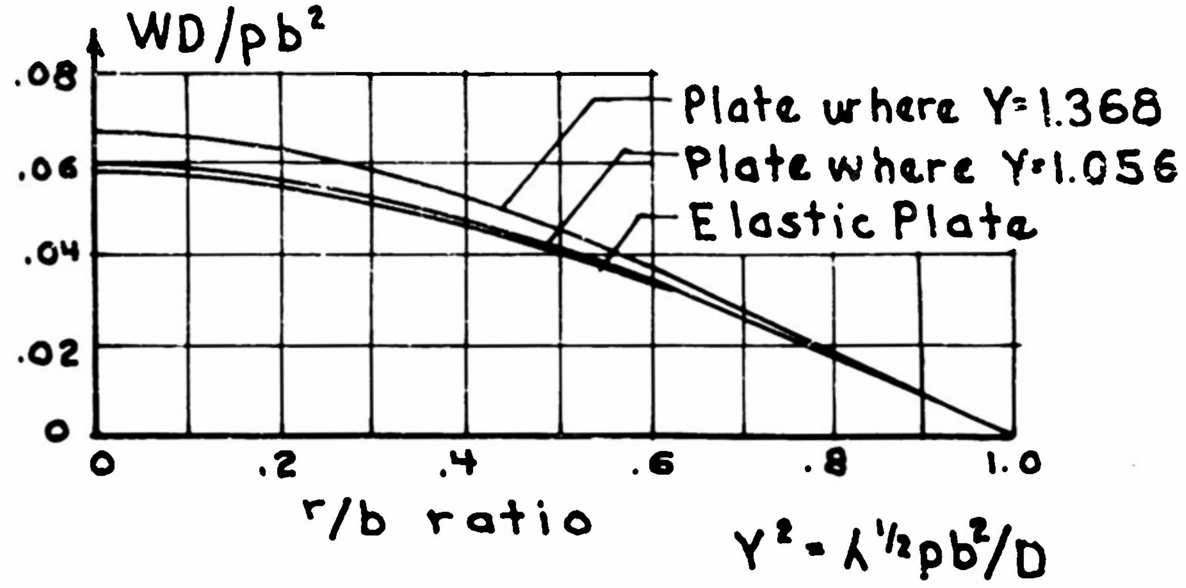


FIGURE 11

Variation of the Radial Bending Moment with the Radial Distance for a Plate where Y=1.368.

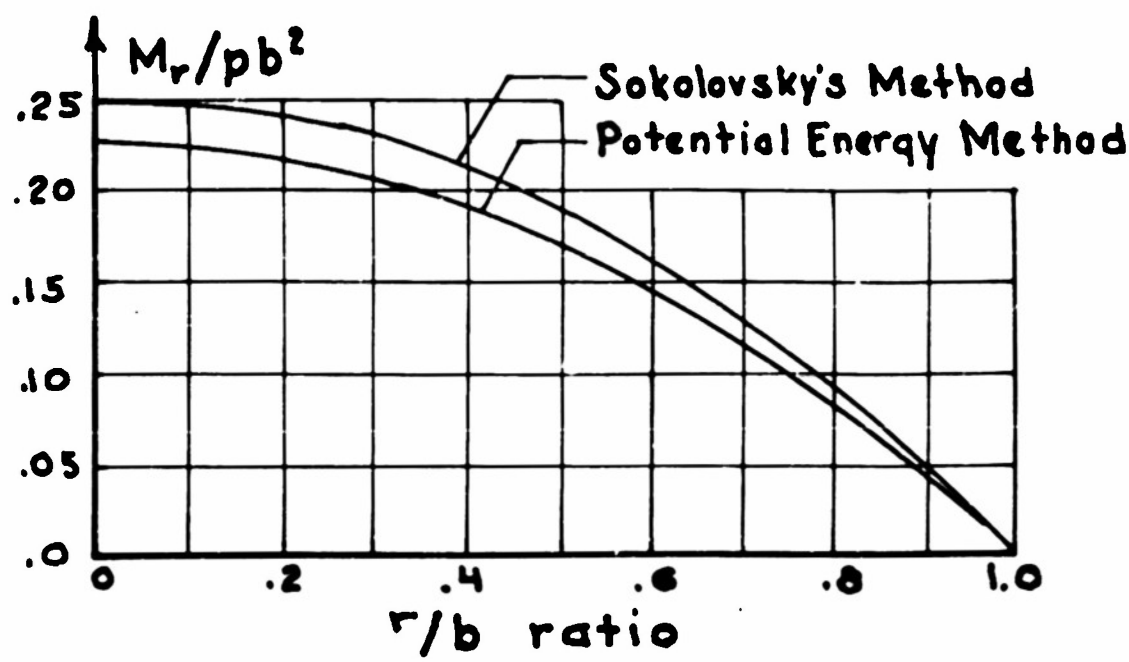


FIGURE 12

A Graphical Representation of the Stress-Strain Relations

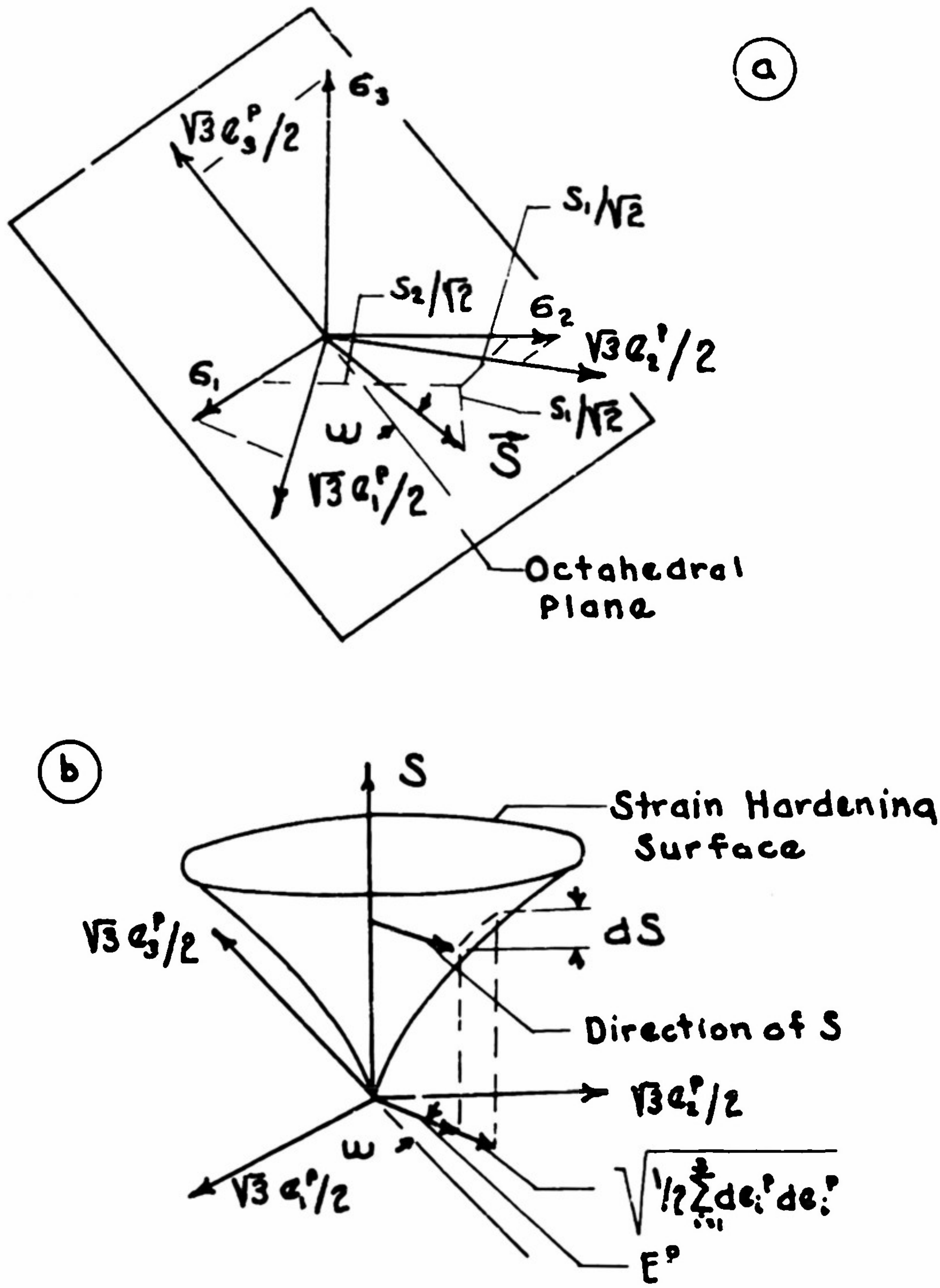


FIGURE 13